VANISHING OF EXT AND TOR OVER SOME COHEN-MACAULAY LOCAL RINGS

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Abstract. We discuss vanishing of cohomology of finite modules over Cohen-Macaulay local rings $(R, \mathfrak{m})$. Special attention is given to the case when the modules are annihilated by $\mathfrak{m}^2$. (Note that if $\mathfrak{m}^3 = 0$, then we can assume the modules satisfy this condition.) In this case we obtain effective versions of conjectures of Auslander-Reiten and Tachikawa.

Introduction

In recent years, there has been growing interest in understanding the vanishing of Ext and Tor over Noetherian local rings, especially in the case when the ring is Artinian. One motivation for this interest comes from a conjecture of Auslander and Reiten [2], which in the case of commutative local rings can be stated as follows:

Conjecture (Auslander-Reiten). *Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, and M a finitely generated R-module. If $\mathrm{Ext}^i_R(M, M \oplus R) = 0$ for all $i > 0$, then M is free.*

This conjecture was initially stated for Artin algebras, and Auslander, Ding and Solberg [1] widened the context to algebras over commutative local rings. A recent result of Huneke and Leuschke [13] establishes the conjecture in the case when R is a quotient by a regular sequence of an excellent Cohen-Macaulay normal domain containing the rational numbers.

To prove the Auslander-Reiten Conjecture for Cohen-Macaulay rings, it suffices to consider Artinian rings. Indeed, if R is assumed to be Cohen-Macaulay, then one can first replace M by a high syzygy in a free resolution of M (see [2]) to assume that M is maximal Cohen-Macaulay. Now, if $x_1, ..., x_d$ is a maximal M- and R-regular sequence, and I is the ideal generated by it, then the vanishing of $\mathrm{Ext}_R(M, M \oplus R)$ passes to $\mathrm{Ext}_{R/I}(M/IM, M/IM \oplus R/I)$; also, M has finite projective dimension (and it is thus free) if and only if M/IM has. Thus, replacing R by R/I and M by M/IM, one can assume without loss of generality that R is Artinian.

In this paper we concentrate on the commutative Artinian case. If $\mathfrak{m}^2 = 0$, then the first syzygy in a minimal free resolution of any non-free R-module is annihilated by the maximal ideal, and the Auslander-Reiten conjecture follows trivially. The first interesting open case is when $\mathfrak{m}^3 = 0$.

Rings in which $\mathfrak{m}^3 = 0$ were systematically studied by Lescot [15]. In particular, his results give the Poincaré series of finitely generated modules none of whose syzygies has a copy of the residue field as a direct summand. Only such

Date: September 22, 2003.
C.H. and A.N.V. were partially supported by NSF grant DMS-0098654.
L.M.Ş. was supported by Clay Mathematics Institute during the Summer 2002.

modules could provide counterexamples to the Auslander-Reiten conjecture. For if $\mathrm{Ext}^i_R(M, M \oplus R) = 0$ for all $i > 0$, and M is not free, it is clear by shifting degree that no syzygy of M can have the residue field as a direct summand.

One of our main results proves the Auslander-Reiten conjecture for rings with $\mathfrak{m}^3 = 0$. Note that the statement gives an effective bound on the required number of vanishing Ext modules.

4.1. **Theorem.** *Let $(R, \mathfrak{m})$ be a commutative Artinian local ring with $\mathfrak{m}^3 = 0$ and M a finitely generated R-module.*

(1) *If $\mathrm{Ext}^i_R(M, M \oplus R) = 0$ for four consecutive values of i with $i \geq 2$, then M is free.*
(2) *If R is Gorenstein and $\mathrm{Ext}^i_R(M, M) = 0$ for some $i > 0$, then M is free.*

The case $i = 1$ in the second part is known: Hoshino [12] proved, more generally, that if R is a (possibly non-commutative) finite dimensional self-injective local algebra with radical cube zero and $\mathrm{Ext}^1_R(M, M) = 0$, then M is free.

As mentioned above, in the context of the Auslander-Reiten conjecture one can replace the original module M by an arbitrary syzygy in a minimal free resolution of M. Now, if $\mathfrak{m}^3 = 0$ and N is such a syzygy, then $\mathfrak{m}^2 N = 0$. A closer look at modules annihilated by $\mathfrak{m}^2$ shows that the Auslander-Reiten conjecture holds for any such module over an arbitrary Artinian local ring. We also give a bound on the required number of vanishing Exts, in terms of the minimal number of generators, denoted $\nu(-)$, of certain modules.

4.2. **Theorem.** *Let $(R, \mathfrak{m})$ be a commutative Artinian local ring and M a finitely generated R-module with $\mathfrak{m}^2 M = 0$.*

If $\mathrm{Ext}^i_R(M, M \oplus R) = 0$ for all i with $0 < i \leq \max\{3, \nu(M), \nu(\mathfrak{m}M)\}$, then M is free.

We prove our results by considering more generally the vanishing of $\mathrm{Ext}_R(M, N)$ where M and N are two finitely generated modules over an Artinian ring R. Since the Matlis dual of such Ext modules are Tor modules, we often find it more convenient to work with the vanishing properties of Tor. Our arguments suggest that the vanishing of $\mathrm{Tor}^R_i(M, N)$ for all positive i places restrictions which relate the annihilators of M, N and R. Specifically, we propose the following:

5.1. **Conjecture.** *Let $(R, \mathfrak{m})$ be a commutative Artinian local ring and let M, N be nonzero finitely generated R-modules with $\mathfrak{m}^2 M = \mathfrak{m}^2 N = 0$. If $\mathrm{Tor}^R_i(M, N) = 0$ for all $i > 0$, then $\mathfrak{m}^3 = 0$.*

One can ask a more general question: Let p, q be positive integers and assume that M, N are nonzero modules with $\mathfrak{m}^p M = \mathfrak{m}^q N = 0$. If $\mathrm{Tor}^R_i(M, N) = 0$ for all $i > 0$, does it follow that $\mathfrak{m}^{p+q-1} = 0$?

The hypothesis of Conjecture 5.1 imposes a strong condition on the Poincaré series of the residue field of R, cf. Lemma 1.8. This allows us to verify the conjecture for several classes of rings, including complete intersections of codimension greater than 2, Koszul rings, Golod rings, etc. In Theorem 5.4 we prove the conjecture when the ring R is standard graded.

Another conjecture which has received attention recently is a conjecture of Tachikawa. A commutative version of this conjecture for Cohen-Macaulay local rings is the following, cf. Avramov, Buchweitz and Şega [6], and also Hanes and Huneke [11]:

Conjecture (Tachikawa). *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring. If R has a canonical module ω and $\mathrm{Ext}^i_R(\omega, R) = 0$ for all $i > 0$, then R is Gorenstein, i.e. ω is free.*

This version of the Tachikawa conjecture is subsumed by the Auslander-Reiten conjecture, since the condition $\mathrm{Ext}^i_R(\omega, \omega) = 0$ for all $i > 0$ holds automatically when R is Cohen-Macaulay.

Assuming that $\mathfrak{m}^3 = 0$ and R is a finite dimensional local algebra over a field, this conjecture was proved by Asashiba [3] under the weaker assumption that $\mathrm{Ext}^1_R(\omega, R) = 0$.

Recall that $\mathrm{Ext}^i_R(\omega, R)$ is Matlis dual to $\mathrm{Tor}^R_i(\omega, \omega)$ when R is Artinian. Thus, the theorem below is equivalent to the one that appears in Section 2 under the same number.

2.11. **Theorem.** *Let $(R, \mathfrak{m})$ be a commutative Artinian local ring with $\mathfrak{m}^3 = 0$. The following statements are equivalent:*

(1) *R is Gorenstein.*
(2) $\mathrm{Ext}^1_R(\omega, R) = 0$.
(3) $\mathrm{Ext}^2_R(\omega, R) = \mathrm{Ext}^3_R(\omega, R) = 0$.
(4) $\mathrm{Ext}^j_R(\omega, R) = \mathrm{Ext}^{j+1}_R(\omega, R) = \mathrm{Ext}^{j+2}_R(\omega, R) = 0$ *for some $j \geq 0$.*

A different proof of the equivalence (1) $\iff$ (2), along the lines of [4], is given in [6].

Section 1 contains a number of lemmas we will use throughout the paper, concerning the growth of Betti numbers of modules under certain conditions on the vanishing of Tor.

Section 2 contains our work on rings with $\mathfrak{m}^3 = 0$. An important technical result is Theorem 2.5, which gives detailed information comparing the Betti numbers of two modules with three consecutive vanishing Tors. In this section we prove Theorem 2.11.

In Section 3 we show that if sufficiently many Tors vanish for two modules M,N and $\mathfrak{m}^2 M = 0$, then either M or N is free. However, for this result we impose stringent conditions on the ring.

Section 4 contains the proof of Theorems 4.1 and 4.2.

In Section 5 we deal with Conjecture 5.1, and give the proof of Theorem 5.4.

In Section 6 we prove Tachikawa's conjecture for Cohen-Macaulay rings of type two.

1. Betti numbers

In this paper $(R, \mathfrak{m}, k)$ denotes a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. We consider finitely generated R-modules M, N.

The number $\nu(M)$ denotes the minimal number of generators of M and $\lambda(M)$ denotes the length of M.

For every $i \geq 0$ we let M_i denote the i-th syzygy of M in a minimal free resolution

$$\ldots \longrightarrow R^{b_{i+1}(M)} \xrightarrow{\delta_i} R^{b_i(M)} \longrightarrow \ldots \xrightarrow{\delta_0} R^{b_0(M)} \longrightarrow M \longrightarrow 0$$

The number $b_i(M)$ is called the i-th *Betti number* of M over R. The *Poincaré series* of M over R is the formal power series

$$P_M^R(t) = \sum_{i=0}^{\infty} b_i(M)t^i$$

In this section we describe several restrictions on the Betti numbers of M,N that are imposed under the assumption that $\operatorname{Tor}_i^R(M,N)=0$ for certain values of i.

1.1. *Remark.* If $\operatorname{Tor}_i^R(M,N)=0$ for some $i>0$ and N has infinite projective dimension, then k is not a direct summand of any of the R-modules $M_0,\dots,M_{i-1}$. Indeed, we have $\operatorname{Tor}_{i-j}^R(M_j,N)\cong \operatorname{Tor}_i^R(M,N)=0$ for all $j<i$. If k is a direct summand of M_j for some $j<i$, then $\operatorname{Tor}_{i-j}^R(k,N)=0$, contradicting the assumption on N.

Let $P(t)=a_0+a_1t+\cdots+a_it^i+\dots$ be a formal power series. For each $n\geq 0$ we denote $[P(t)]_{\leq n}$ the polynomial $a_0+a_1t+\cdots+a_nt^n$.

The next result is a slightly modified version of a technique in [17, 1.1]:

1.2**. Lemma.** *Let n be a positive integer. If $\operatorname{Tor}_i^R(M,N)=0$ for all $i\in[1,n]$, then*

$$\left[P^R_{M\otimes_R N}(t)\right]_{\leq n} = \left[P_M^R(t)P_N^R(t)\right]_{\leq n}$$

Proof. Let F, respectively G, be a minimal free resolution of M, respectively N, over R. The hypothesis implies that $\mathrm{H}_i(F\otimes_R G)=0$ for all $i\in[1,n]$. Since $F\otimes_R G$ is a minimal complex with $\mathrm{H}_0(F\otimes_R G)=M\otimes_R N$, we note that $(F\otimes_R G)_{\leq n}$ is the beginning of a minimal free resolution of $M\otimes_R N$. We have thus:

$$[P^R_{M\otimes_R N}]_{\leq n} = \sum_{i=0}^{n}\operatorname{rank}(F\otimes_R G)_i t^i = [P_M^R(t)P_N^R(t)]_{\leq n} \qquad \square$$

1.3**.** For each nonzero R-module M of finite length we set

$$\gamma(M)=\frac{\lambda(M)}{\nu(M)}-1$$

Note that $\gamma(M)$ is also equal to $\dfrac{\lambda(\mathfrak{m}M)}{\nu(M)}$. It is thus a rational number in the interval $[0,\lambda(R)-1]$. The extreme values on this interval are attained as follows: $\gamma(M)=0$ if and only if $\mathfrak{m}M=0$ and $\gamma(M)=\lambda(R)-1$ if and only if M is free.

We will often use the definition of $\gamma(M)$ in length computations as follows:

$$\lambda(M)=\nu(M)\big(\gamma(M)+1\big) \tag{1.3.1}$$

1.4**. Lemma.** *Let R be an Artinian ring and let M, N be finiteley generated R-modules such that M is not zero and N is not free.*

(1) *If $\operatorname{Tor}_i^R(M,N)=0$ for some $i>0$, then*

$$\big(\gamma(M\otimes_R N_i)+1\big)b_i(N)=\big(\gamma(M)-\gamma(M\otimes_R N_{i-1})\big)b_{i-1}(N)$$

In particular, there is an inequality $b_i(N)\leq\gamma(M)b_{i-1}(N)$.

(2) *If $\mathfrak{m}^2M=0$ and $\operatorname{Tor}_i^R(M,N)=0$ for some $i>0$, then $\mathfrak{m}(M\otimes_R N_i)=0$ and*

$$b_i(N)=\big(\gamma(N)-\gamma(M\otimes_R N_{i-1})\big)b_{i-1}(N)$$

(3) *If $\mathfrak{m}^2M=0$ and $\operatorname{Tor}_i^R(M,N)=\operatorname{Tor}_{i-1}^R(M,N)=0$ for some $i>1$, then*

$$b_i(N)=\gamma(M)b_{i-1}(N)$$

Proof. (1) Consider the short exact sequence

$$0 \longrightarrow N_i \longrightarrow R^{b_{i-1}(N)} \longrightarrow N_{i-1} \longrightarrow 0$$

The hypothesis implies that the sequence remains exact when tensored with M:

$$0 \longrightarrow M \otimes_R N_i \longrightarrow M \otimes_R R^{b_{i-1}(N)} \longrightarrow M \otimes_R N_{i-1} \longrightarrow 0 \tag{1.4.1}$$

For any j we use (1.3.1) to obtain

$$\lambda(M \otimes_R N_j) = \nu(M \otimes_R N_j)\big(\gamma(M \otimes_R N_j) + 1\big) = \nu(M) b_j(N)\big(\gamma(M \otimes_R N_j) + 1\big)$$
$$\lambda(M \otimes_R R^{b_j(N)}) = b_j(N)\lambda(M) = b_j(N)\nu(M)\big(\gamma(M) + 1\big)$$

Using these expressions, a length count in (1.4.1) leads to the desired conclusion.

(2) As above, we have a short exact sequence (1.4.1). The image of N_i in $R^{b_{i-1}(N)}$ is contained in $\mathfrak{m}R^{b_{i-1}(N)}$, hence the image of $M \otimes_R N_i$ in $M \otimes_R R^{b_{i-1}(N)}$ is contained in $\mathfrak{m}(M \otimes_R R^{b_{i-1}(N)})$, and the latter is annihilated by $\mathfrak{m}$. We have then $\gamma(M \otimes_R N_i) = 0$ and the relation follows from (1).

(3) By (2) we have $\mathfrak{m}(M \otimes_R N_{i-1}) = \mathfrak{m}(M \otimes_R N_i) = 0$ and therefore $\gamma(M \otimes_R N_{i-1}) = \gamma(M \otimes_R N_i) = 0$. □

1.5. **Lemma.** *Let R be an Artinian ring and let M, N be finitely generated R-modules such that M is not zero and N is not free.*

(1) *If $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \in [1, \nu(N)]$, then $\gamma(M) \geq 1$.*
(2) *If $\mathfrak{m}^2 M = 0$ and $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \in [1, \log_2 b_1(N) + 2]$, then $\gamma(M)$ is an integer.*

Proof. (1) Assume that $\gamma(M) < 1$. By Lemma 1.4(1) we have then

$$b_i(N) \leq \gamma(M) b_{i-1}(N) < b_{i-1}(N) \quad \text{for all} \quad i \in [1, \nu(N)]$$

Since $b_0(N) = \nu(N)$ we conclude that $b_{\nu(N)}(N) = 0$, hence N has finite projective dimension and it is thus free, contradicting the hypothesis.

(2) Using Lemma 1.4(3) we have:

$$b_{i+1}(N) = (\gamma(M))^i b_1(N) \quad \text{for all} \quad i \in [1, \log_2 b_1(N) + 1]$$

Let u, v be relatively prime positive integers such that $\gamma(M) = uv^{-1}$. It follows that v^i divides $b_1(N)$ for all $i \in [1, \log_2 b_1(N) + 1]$. If $v \geq 2$, then $b_1(N) \geq 2^i$ for all such i, a contradiction. □

Let e denote the minimal number of generators of $\mathfrak{m}$.

1.6. **Lemma.** *Let R be an Artinian ring, and let M, N be non-free finitely generated R-modules. If $\mathfrak{m}^2 M = 0$ and $\operatorname{Tor}_2^R(M, N) = \operatorname{Tor}_1^R(M, N) = 0$, then the following hold:*

(1) *$b_1(M) = \big(e - \gamma(M)\big) b_0(M)$.*
(2) *$\mathfrak{m} M_1 = \mathfrak{m}^2 R^{b_0(M)}$.*

Proof. (1) Lemma 1.4(2) shows that the R-module $M \otimes_R N_1$ is a finite direct sum of copies of k, hence its first Betti number is $e b_0(M) b_1(N)$. On the other hand, since $\operatorname{Tor}_1(M, N_1) = 0$, Lemma 1.2 gives $b_1(M \otimes_R N_1) = b_0(M) b_2(N) + b_1(M) b_1(N)$. We also have $b_2(N) = \gamma(M) b_1(N)$ by Lemma 1.4(3), hence

$$e b_0(M) b_1(N) = b_0(M) \gamma(M) b_1(N) + b_1(M) b_1(N)$$

(2) A length count in the short exact sequence

$$0 \longrightarrow M_1 \longrightarrow R^{b_0(M)} \longrightarrow M \longrightarrow 0$$

using (1.3.1) gives

$$\begin{aligned}\lambda(M_1) &= \lambda(R)b_0(M) - \lambda(M)\\ &= \big(1+e+\lambda(\mathfrak{m}^2)\big)b_0(M) - b_0(M)\big(\gamma(M)+1\big)\\ &= \big(e+\lambda(\mathfrak{m}^2)-\gamma(M)\big)b_0(M)\end{aligned}$$

We next use (1) to obtain

$$\begin{aligned}\lambda(\mathfrak{m}M_1) &= \lambda(M_1) - \nu(M_1) = \lambda(M_1) - b_1(M)\\ &= \big(e+\lambda(\mathfrak{m}^2)-\gamma(M)\big)b_0(M) - \big(e-\gamma(M)\big)b_0(M)\\ &= \lambda(\mathfrak{m}^2)b_0(M)\end{aligned}$$

Since $\mathfrak{m}M_1$ is contained in $\mathfrak{m}^2R^{b_0(M)}$ and both modules have the same length, it follows that they are equal. ☐

1.7**. Lemma.** *Let R be an Artinian ring and let M, N be non-free finite R-modules with $\mathfrak{m}^2M = \mathfrak{m}^2N = 0$.*

If $\operatorname{Tor}_2^R(M,N) = \operatorname{Tor}_1^R(M,N) = 0$, then $\gamma(M)+\gamma(N)-\gamma(M\otimes_R N) = e$

Proof. Compare the relations

$$b_1(M) = \big(\gamma(N)-\gamma(M\otimes_R N)\big)b_0(M) \quad \text{and} \quad b_1(M) = \big(e-\gamma(M)\big)b_0(M)$$

given by Lemma 1.4(2), respectively Lemma 1.6(1). ☐

1.8**. Lemma.** *Let R be an Artinian ring and let M, N be finitely generated non-free R-modules with $\mathfrak{m}^2M = \mathfrak{m}^2N = 0$.*

If $\operatorname{Tor}_i^R(M,N) = 0$ for all $i > 0$, then

$$P_k^R(t) = \frac{1-\gamma(M\otimes_R N)t}{\big(1-\gamma(M)t\big)\big(1-\gamma(N)t\big)}$$

Proof. By Lemma 1.4 we have

$$P_{N_1}^R(t) = \frac{b_1(N)}{\big(1-\gamma(M)t\big)} \quad \text{and} \quad P_M^R(t) = b_0(M) + \frac{b_0(M)\big(\gamma(N)-\gamma(M\otimes_R N)\big)t}{\big(1-\gamma(N)t\big)}$$

Lemma 1.2 then yields

$$P_{M\otimes_R N_1}^R(t) = P_M^R(t)P_{N_1}^R(t) = b_0(M)b_1(N)\frac{1-\gamma(M\otimes_R N)t}{\big(1-\gamma(M)t\big)\big(1-\gamma(N)t\big)}$$

The desired conclusion about $P_k^R(t)$ is then obtained using the fact that $\mathfrak{m}(M\otimes_R N_1) = 0$, cf. Lemma 1.4(2). ☐

2. Rings with $\mathfrak{m}^3 = 0$

In this section, unless otherwise stated, we assume that $(R,\mathfrak{m},k)$ is an Artinian local ring with $\mathfrak{m}^3 = 0$. We set $e = \nu(\mathfrak{m})$ and $a = \dim_k \operatorname{Soc}(R)$.

When $\mathfrak{m}^2 = 0$, vanishing of homology is not at all mysterious:

2.1. *Remark.* If $\mathfrak{m}^2 = 0$ and $\operatorname{Tor}_i^R(M,N) = 0$ for some $i > 1$, then M or N is free.

Indeed, assume that M is not free. The module M_1 is contained in $\mathfrak{m}R^{b_0(M)}$, hence $\mathfrak{m}M_1 = 0$. It is thus a finite sum of copies of k. Since $\operatorname{Tor}_{i-1}^R(M_1,N) = 0$, we conclude that N has finite projective dimension, hence it is free.

The behavior of Betti numbers of finitely generated R-modules was studied by Lescot [15]. The following results are collected from the proof of [15, 3.3].

2.2. Assume M is not free. For any $i \geq 0$ the following hold:

(1) There is an inequality $b_{i+1}(M) \geq e b_i(M) - \nu(\mathfrak{m}M_i)$. Equality holds if and only if k is not a direct summand of M_{i+1}.
(2) If $i > 1$ and k is not a direct summand of M_i, then $\nu(\mathfrak{m}M_i) = a b_{i-1}(M)$.

2.3. *Remark.* If $\mathfrak{m}^2 M = 0$ and k is not a direct summand of M, then $\operatorname{Soc}(M) = \mathfrak{m}M$. (This statement holds for all local rings R, not only for those with $\mathfrak{m}^3 = 0$.)

2.4. *Remark.* If $\operatorname{Tor}_i^R(M,N) = 0$ for some $i \geq 3$ and M, N are not free, then $\operatorname{Soc}(R) = \mathfrak{m}^2$. Indeed, it is enough to show that $\operatorname{Soc}(R) \subseteq \mathfrak{m}^2$, the other inclusion being obvious. Note that $\operatorname{Soc}(M_{i-1}) = \operatorname{Soc}(R^{b_{i-2}(M)})$. On the other hand, Remark 1.1 shows that k is not a direct summand in M_{i-1}, hence $\operatorname{Soc}(M_{i-1}) = \mathfrak{m}M_{i-1} \subseteq \mathfrak{m}^2 R^{b_{i-2}(M)}$ by Remark 2.3, and the conclusion follows.

2.5. **Theorem.** *Let $(R,\mathfrak{m})$ be an Artinian local ring with $\mathfrak{m}^3 = 0$, and let M,N be non-free R-modules satisfying $\mathfrak{m}^2 M = \mathfrak{m}^2 N = 0$.*

If there exists an integer $j > 0$ such that

$$\operatorname{Tor}_j^R(M,N) = \operatorname{Tor}_{j+1}^R(M,N) = \operatorname{Tor}_{j+2}^R(M,N) = 0$$

then the following hold:

(1) *$\gamma(M)$ and $\gamma(N)$ are positive integers.*
(2) $\dfrac{b_{i+1}(M)}{b_i(M)} = \gamma(N)$ *and* $\dfrac{b_{i+1}(N)}{b_i(N)} = \gamma(M)$ *for all i with $0 \leq i \leq j+1$.*
(3) *$\gamma(M) = \gamma(M_i)$ and $\gamma(N) = \gamma(N_i)$ for all i with $0 \leq i \leq j$.*
(4) *$\gamma(M) + \gamma(N) = e$ and $\gamma(M)\gamma(N) = a$.*

2.6. *Remark.* It is not true that the vanishing of $\operatorname{Tor}_i^R(M,N)$ for all $i \gg 0$ implies $\operatorname{Tor}_i^R(M,N) = 0$ for all $i > 0$. Recently, Jorgensen and Şega [14] have constructed a local ring R with $\mathfrak{m}^3 = 0$ and an R-module M such that for every integer n there exists an R-module N_n satisfying $\operatorname{Tor}_i^R(M,N_n) = 0$ if and only if $i \neq 0, n, n+1$.

Proof of Theorem 2.5. (1) We will show that $\gamma(M)$, $\gamma(N)$ satisfy the equation $\gamma^2 - e\gamma + a = 0$. As $\gamma(M)$, $\gamma(N)$ are positive rational numbers, this implies that they are integers. The statement is symmetric in M and N, hence it suffices to prove it for $\gamma(M)$.

By Lemma 1.4 we have:

(2.6.1) $\quad b_j(N) \leq \gamma(M) b_{j-1}(N) \quad$ and $\quad b_{i+1}(N) = \gamma(M) b_i(N) \quad$ for $\quad i = j, j+1$

The hypothesis and Remark 1.1 imply that k is not a direct summand of M_i for any i with $0 \leq i \leq j+1$ and then 2.2 gives the following relations:

(2.6.2) $\quad b_{j+1}(N) = e b_j(N) - a b_{j-1}(N) \quad$ and $\quad b_{j+2}(N) \geq e b_{j+1}(N) - a b_j(N)$

Combining the second relations of (2.6.1) and (2.6.2) we obtain

$$\gamma(M)^2 b_j(N) \geq e\gamma(M) b_j(N) - a b_j(N)$$

Canceling $b_j(N)$ we get $\gamma(M)^2 \geq e\gamma(M) - a$.

On the other hand, using the first relation of (2.6.2) and (2.6.1) we have:

$$\gamma(M)b_j(N) = eb_j(N) - ab_{j-1}(N) \leq eb_j(N) - a\frac{b_j(N)}{\gamma(M)}$$

Canceling $b_j(N)$ and multiplying both sides by $\gamma(M)$ we obtain $\gamma(M)^2 \leq e\gamma(M) - a$.

We conclude:

$$\gamma(M)^2 - e\gamma(M) + a = 0 \quad \text{and} \quad \gamma(N)^2 - e\gamma(N) + a = 0 \tag{2.6.3}$$

(2) We show by induction on $j+1-i$ that $b_{i+1}(N) = \gamma(M)b_i(N)$ for all i with $0 \leq i \leq j+1$. By (2.6.1), the relation holds for $i = j, j+1$. Assuming it holds for $i = l+1$, with $0 \leq l < j$, we prove that it holds for $i = l$. By 2.2 we have $b_{l+2}(N) = eb_{l+1}(N) - ab_l(N)$, hence, using the inductive hypothesis and (2.6.3) we obtain

$$ab_l(N) = \big(e - \gamma(M)\big)b_{l+1}(N) = a\gamma(M)^{-1}b_{l+1}(N)$$

and the conclusion follows.

(3) Let i be as in the statement and set $l = j+1-i$. The hypothesis implies $\operatorname{Tor}_l^R(M_i, N) = \operatorname{Tor}_{l+1}^R(M_i, N) = 0$, hence $b_{l+1}(N) = \gamma(M_i)b_l(N)$ by Lemma 1.4(3). By (1), we also have $b_{l+1}(N) = \gamma(M)b_l(N)$, hence $\gamma(M_i) = \gamma(M)$.

(4) By Lemma 1.4(2) we have $\mathfrak{m}(M \otimes_R N_j) = 0$. The hypothesis implies $\operatorname{Tor}_1(M, N_j) = \operatorname{Tor}_2(M, N_j) = 0$, hence, by Lemma 1.7 we get $\gamma(M) + \gamma(N_j) = e + \gamma(M \otimes_R N_j) = e$. Using (2) we have then $\gamma(M) + \gamma(N) = e$. Recall from (2.6.3) that $\gamma(M)$ and $\gamma(N)$ are roots for the equation $\gamma^2 - e\gamma + a = 0$. We obtain:

$$\begin{aligned} 2\gamma(M)\gamma(N) &= \big(\gamma(M) + \gamma(N)\big)^2 - \gamma(M)^2 - \gamma(N)^2 \\ &= e^2 - \big(e\gamma(M) - a\big) - \big(e\gamma(N) - a\big) \\ &= e^2 - e\big(\gamma(M) + \gamma(N)\big) + 2a = 2a \end{aligned}$$

and this finishes the proof of the theorem. □

Remark. Let M, N, j be as in the statement of Theorem 2.5. If $l \geq j+3$ and k is not a direct summand of M_i for all $i < l$ (In view of Lemma 1.1 this happens, for example, when $\operatorname{Tor}_l^R(M, N) = 0$), then

$$\frac{b_{i+1}(M)}{b_i(M)} = \gamma(N) \quad \text{and} \quad \frac{b_{i+1}(N)}{b_i(N)} = \gamma(M) \quad \text{for all } i \text{ with} \quad 0 \leq i \leq l-1$$

Indeed, by 2.2 we have $b_{i+1}(N) = eb_i(N) - ab_{i-1}(N)$ for all $i \leq l-2$. We proceed by induction on i, as in the proof of Theorem 2.5. By this theorem, the statement is true for all $i \leq j+1$. Assuming that $i \leq l-1$ and $b_i(N) = \gamma(N)b_{i-1}(N)$, we then have:

$$b_{i+1}(N) = eb_i(N) - a\frac{b_i(N)}{\gamma(M)} = b_i(N)\Big(e - \frac{a}{\gamma(M)}\Big) = b_i(N)\gamma(M)$$

where the last inequality is due to (2.6.3).

Since R is Artinian, it has a dualizing module ω. In the remaining part of the section we present results that are obtained when one of the modules M, N is equal to ω. An important part in our arguments is played by Matlis duality. We recall below some basic facts.

2.7. Let R be an Artinian ring (not necessarily with $\mathfrak{m}^3 = 0$) and let ω denote its dualizing module. Matlis duality then gives: $\nu(\omega) = \dim_k \operatorname{Soc}(R)$, $\dim_k \operatorname{Soc}(\omega) = 1$ and $\lambda(\omega) = \lambda(R)$.

2.7.1. For every R-module M we set $M^\vee = \operatorname{Hom}_R(M, \omega)$. For any R-modules M, N and any i there are isomorphisms:

$$\operatorname{Tor}_i^R(M, N^\vee) \cong \operatorname{Ext}_R^i(M, N)^\vee$$

2.7.2. We set $M^* = \operatorname{Hom}_R(M, R)$. Note that $M \otimes_R \omega \cong (M^*)^\vee$. Indeed, this is given by Matlis duality and the isomorphisms below:

$$(M \otimes_R \omega)^\vee \cong \operatorname{Hom}_R(M \otimes_R \omega, \omega) \cong \operatorname{Hom}_R\big(M, \operatorname{Hom}_R(\omega, \omega)\big) \cong M^*$$

In particular, we obtain that R is Gorenstein if and only if ω is isomorphic to ω^{**}. Indeed, if R is Gorenstein, then the relation holds trivially. Conversely, if $\omega^{**} \cong \omega$, then, taking $M = \omega^*$ above, we obtain

$$\omega^* \otimes_R \omega \cong (\omega^{**})^\vee \cong \omega^\vee \cong R$$

and it follows that ω is cyclic, hence R is Gorenstein.

We now return to the case of interest, when $\mathfrak{m}^3 = 0$.

2.8. Assume that $\mathfrak{m}^2 \neq 0$. By the above, we have $\nu(\omega) = a$. Since $\mathfrak{m}^2\omega$ is not zero and is contained in $\operatorname{Soc}(\omega)$, we also have $\nu(\mathfrak{m}^2\omega) = 1$. Setting $N = \omega_1$ and $r = \nu(\mathfrak{m}^2)$, we can make then the following computations:

(1) $\lambda(\omega) = \lambda(R) = 1 + r + e$.
(2) $\nu(\mathfrak{m}\omega) = \lambda(\omega) - \nu(\mathfrak{m}^2\omega) - \nu(\omega) = 1 + r + e - 1 - a = e + r - a$.
(3) $\lambda(N) = (a-1)(1 + r + e)$. This follows from a length count in the short exact sequence

$$0 \longrightarrow N \longrightarrow R^a \longrightarrow \omega \longrightarrow 0$$

(4) If k is not a direct summand of N and $a = r$, then 2.2(1) and (2) give

$$\nu(N) = e\nu(\omega) - \nu(\mathfrak{m}\omega) = ea - e = e(a-1)$$
$$\gamma(N) = \frac{\lambda(N)}{\nu(N)} - 1 = \frac{(a-1)(1+a+e)}{(a-1)e} - 1 = \frac{1+a}{e}$$

2.9. **Proposition.** *Let $(R, \mathfrak{m})$ be an non-Gorenstein Artinian ring with $\mathfrak{m}^3 = 0$ and M a non-free finitely generated R-module with $\mathfrak{m}^2 M = 0$.*

If there exists an integer $j \geq 2$ such that

$$\operatorname{Tor}_j^R(M, \omega) = \operatorname{Tor}_{j+1}^R(M, \omega) = \operatorname{Tor}_{j+2}^R(M, \omega) = 0$$

then $e = a + 1$, $\gamma(\omega_1) = 1$, $\gamma(M) = a$ and $b_0(M) = b_1(M) = \cdots = b_{j+2}(M)$.

Proof. Set $N = \omega_1$. By Remark 2.4 we have $\operatorname{Soc}(R) = \mathfrak{m}^2$. Also, k is not a direct summand of N by Remark 1.1, hence 2.8(4) gives $\gamma(N) = (1 + a)/e$.

By Theorem 2.5(4), $\gamma(N)$ is a solution of the equation $\gamma^2 - e\gamma + a = 0$. It follows that $(a + 1)^2 = e^2$, hence $a + 1 = e$. In particular, $\gamma(N) = 1$, and Theorem 2.5(4) implies $\gamma(M) = a$. The conclusion about the Betti numbers follows from Theorem 2.5(2). $\square$

Proposition 2.9 is related to a result of Yoshino [21, 3.1].

There are examples when the situation in Proposition 2.9 holds, M is not free, and j can be chosen to be arbitrarily large. Such an example is provided by Avramov, Gasharov and Peeva [7, 2.2], as described below.

If F is a complex, then F^* denotes the complex $\operatorname{Hom}_R(F, R)$, with induced differentials.

2.10. **Example.** Let l be a field, let $X = \{X_1, X_2, X_3, X_4\}$ be a set of indeterminates over l and set $A = l[X]_{(X)}$. Let I be the ideal of A generated by elements

$$X_1^2, X_1X_2 - X_3X_4, X_1X_2 - X_4^2, X_1X_3 - X_2X_4, X_1X_4 - X_2^2,$$
$$X_1X_4 - X_2X_3, X_1X_4 - X_3^2$$

and set $R = A/I$. The ring R is then local and has $\mathfrak{m}^3 = 0$.

Let x_i denote the image of X_i in R for $i = 1, \dots, 4$ and consider the sequence of homomorphisms of free R-modules:

$$F = \qquad \dots \xrightarrow{\psi} R^2 \xrightarrow{\varphi} R^2 \xrightarrow{\psi} R^2 \xrightarrow{\varphi} \dots$$

where

$$\varphi = \begin{pmatrix} x_3 & x_1 \\ x_4 & x_2 \end{pmatrix} \qquad \psi = \begin{pmatrix} x_2 & -x_1 \\ -x_4 & x_3 \end{pmatrix}$$

Set $M = \operatorname{Coker}\varphi$. By [7, (2.2)(i)] the complex F is exact. As noted by Veliche [20], a computation similar to one in [7, Section 3] shows that the complex F^* is exact. This yields $\operatorname{Ext}_R^i(M, R) = 0$ for all $i > 0$, or equivalently, cf. 2.7.1, $\operatorname{Tor}_i^R(M, \omega) = 0$ for all $i > 0$.

2.11. **Theorem.** *Let $(R, \mathfrak{m})$ be an Artinian local ring with $\mathfrak{m}^3 = 0$. The following statements are equivalent:*

(1) *R is Gorenstein.*
(2) $\operatorname{Tor}_1^R(\omega, \omega) = 0$.
(3) $\operatorname{Tor}_2^R(\omega, \omega) = \operatorname{Tor}_3^R(\omega, \omega) = 0$.
(4) $\operatorname{Tor}_j^R(\omega, \omega) = \operatorname{Tor}_{j+1}^R(\omega, \omega) = \operatorname{Tor}_{j+2}^R(\omega, \omega) = 0$ *for some* $j \geq 0$.

We recall a result of Asashiba and Hoshino [4, 2.1]:

2.12. Let R be a local ring. If M is a faithful R-module and the sequence

$$0 \longrightarrow N \xrightarrow{\varphi} R^2 \xrightarrow{\psi} M \longrightarrow 0$$

is exact, then there exist homomorphisms α and β, and an isomorphism θ making the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\varphi"] \arrow[d, "\alpha"] & R^2 \arrow[r, "\psi"] \arrow[d, "\theta"] & M \arrow[r] \arrow[d, "\beta"'] & 0 \\
0 \arrow[r] & M^* \arrow[r, "\psi^*"] & (R^2)^* \arrow[r, "\varphi^*"] & N^* &
\end{tikzcd}$$

Proof of Theorem 2.11. Set $r = \nu(\mathfrak{m}^2)$ and $N = \omega_1$ and $b_i = b_i(\omega)$.

The implications (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3), (1) $\Rightarrow$ (4) are obvious.

(2) ⇒ (1) Assume R is not Gorenstein. By 2.8 we have $\lambda(\omega) = 1 + e + r$ and $\nu(\mathfrak{m}\omega) = e + r - a$. We obtain then

$$\gamma(\omega) = \frac{1+e+r}{a} - 1 = \frac{1+e+r-a}{a}$$

By 2.2(1) we have $b_1 \geq eb_0 - \nu(\mathfrak{m}\omega) = ea - (e + r - a)$ and by Lemma 1.4 we have $b_1 \leq \gamma(\omega)b_0$. We conclude:

$$ea - (e + r - a) \leq \frac{1+e+r-a}{a} \cdot a = 1 + e + r - a$$

hence $ea - 2e + 2a - 2r \leq 1$, or, equivalently, $e(a - 2) + 2(a - r) \leq 1$. Since $e > 1$ and $a \geq r$ we conclude that $a = 2 = r$ and $\nu(N) = b_1 \geq e$. By 2.8 we have thus

$$\lambda(R) = \lambda(N) = e + 3$$

The hypothesis implies there is a short exact sequence

$$0 \to N \otimes_R \omega \to \omega^2 \to \omega \otimes_R \omega \to 0\,.$$

with $\lambda(\omega^2) = 2(e+3)$, $\lambda(N \otimes_R \omega) = 2\nu(N) + \varepsilon$ and $\lambda(\omega \otimes_R \omega) = 4 + \eta$, where ε and η are nonnegative integers. A length count in the short exact sequence then gives:

$$2e + 6 = 2\nu(N) + \varepsilon + 4 + \eta \geq 2e + 4 + \varepsilon + \eta \qquad (2.10.1)$$

In particular, it follows that $\eta \leq 2$, hence $\lambda(\omega \otimes_R \omega) = 4 + \eta \leq 6$. Note that $\lambda(\omega^*) = \lambda(\omega \otimes_R \omega)$, as we have $\mathrm{Hom}_R(\omega \otimes_R \omega, \omega) \cong \omega^*$.

For the rest of the proof we will look at the commutative diagram in 2.12, with $M = \omega$. Note that $\alpha\colon N \to \omega^*$ is injective. Also, the lower sequence in the diagram is right-exact, by the hypothesis $\mathrm{Ext}^1_R(\omega, R) = 0$. In particular, $\beta\colon \omega \to N^*$ is surjective, or equivalently, the dual map $\beta^\vee\colon (N^*)^\vee \to R$ is injective.

Since N is contained in ω^*, we have $e + 3 = \lambda(N) \leq \lambda(\omega^*) = 4 + \eta$ and thus $e \leq \eta + 1 \leq 3$. In particular, we have $\eta \in \{1, 2\}$.

If $\eta = 1$, then $e = 2$, hence $\lambda(N) = \lambda(\omega^*) = 5$. It follows that α is an isomorphism. The commutative diagram in 2.12 yields that β is an isomorphism, hence $\omega \cong N^* \cong \omega^{**}$. We apply then 2.7.2 to conclude that R is Gorenstein, a contradiction.

If $\eta = 2$, then the inequality (2.10.1) yields $\varepsilon = 0$ and $\nu(N) = 3$, hence $e \leq 3$ and $\lambda(N \otimes_R \omega) = 6$. On the other hand, $N \otimes_R \omega$ is isomorphic to $(N^*)^\vee$ by 2.7.1, and the latter is contained in R, as seen above. Since $\lambda(R) = e + 3 \leq 6$, it follows that $N \otimes_R \omega \cong R$, hence ω is cyclic, contradicting our assumption that R is not Gorenstein.

(3) ⇒ (1) Assume R is not Gorenstein. By Remark 2.4, we have $\mathrm{Soc}(R) = \mathfrak{m}^2$ and then 2.8 gives $\lambda(\omega) = \lambda(R) = 1 + a + e$ and $\nu(\mathfrak{m}\omega) = e$.

Since $\mathrm{Tor}^R_1(N, \omega) = \mathrm{Tor}^R_2(N, \omega) = 0$, we use Theorem 1.4(3) to obtain:

$$b_2 = \gamma(N)b_1 = \frac{\lambda(\mathfrak{m}N)}{\nu(N)}b_1 = \frac{\nu(\mathfrak{m}N)}{b_1}b_1 = \nu(\mathfrak{m}N)$$

On the other hand, 2.2(1) gives $b_2 = eb_1 - \nu(\mathfrak{m}N) = eb_1 - b_2$, hence we have:

$$2b_2 = eb_1 \quad \text{and} \quad \nu(\mathfrak{m}N) = \frac{eb_1}{2}$$

and we conclude $\gamma(N) = e/2$. By Lemma 1.4(1) we then get

$$b_1 \leq \gamma(N)b_0 = \frac{e}{2}a$$

By 2.2(1) we also have $b_1 \geq eb_0 - \nu(\mathfrak{m}\omega) = e(a-1)$. We obtain thus

$$e(a-1) \leq \frac{e}{2}a$$

and we conclude $a \leq 2$. As we assumed R not to be Gorenstein, we have $a = 2$. Recall from 2.8 that $\gamma(N) = (a+1)/e$. Comparing this with the relation $\gamma(N) = e/2$ obtained above, we obtain $e^2 = 6$, a contradiction.

(4) $\Rightarrow$ (1) By the previous implications we may assume $j \geq 2$. Assume that R is not Gorenstein and set $N = \omega_1$. Applying Proposition 2.9 with $M = N$ we obtain $\gamma(N) = 1$. We then use Proposition 2.5(4) with $M = N$ and we conclude $a = \gamma(N)^2 = 1$, hence R is Gorenstein, a contradiction. □

3. Rings of large embedding dimension

For any finitely generated module N we set

$$c(N) = \max\{4, \log_2(b_1(N)) + 2\}$$

(where $\log_2 0 = -\infty$). The *Loewy length* of the ring R, denoted $\ell\ell(R)$, is the smallest integer n for which $\mathfrak{m}^n = 0$.

In this section we prove the following:

3.1. **Theorem.** *Let $(R, \mathfrak{m})$ be an Artinian local ring satisfying $2\nu(\mathfrak{m}) \geq \lambda(R) - \ell\ell(R) + 4$.*

(1) *If $\mathfrak{m}^2 M = 0$ and $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \in [1, c(N)]$, then either M or N is free.*
(2) *If $\mathfrak{m}^3 = 0$ and $\operatorname{Tor}_i^R(M, N) = 0$ for three consecutive values of $i \geq 2$, then either M or N is free.*

3.2. *Remark.* The ring in Example 2.10 has $\nu(\mathfrak{m}) = 4$, $\lambda(R) = 8$ and $\ell\ell(R) = 3$; in particular, the inequality in the statement of Theorem 3.1 is (minimally) not satisfied. This shows that the inequality can not be improved.

Proof of Theorem 3.1. Set $e = \nu(\mathfrak{m})$ and $h = \ell\ell(R) - 1$. Assume that M and N are not free.

(1) Let i be a positive integer. In the proof of [10, 2.2] Gasharov and Peeva show that the following inequality holds for any local Artinian ring R and any finitely generated R-module N.

$$b_{i+1}(N) \geq eb_i(N) - \big(\lambda(\mathfrak{m}^2) + 2 - h\big)b_{i-1}(N)$$

Setting $a = \lambda(\mathfrak{m}^2) + 3 - h$, we conclude that for all positive integers i there is a strict inequality

$$b_{i+1}(N) > eb_i(N) - ab_{i-1}(N)$$

We let then i be any integer such that $2 \leq i \leq c(N) - 2$. Lemma 1.4(3) gives that $b_j(N) = \gamma(M)^{j-1} b_1(N)$, for $j = i, i+1$ and we conclude

$$\gamma(M)^2 - e\gamma(M) + a > 0 \tag{3.2.1}$$

The roots of the equation $\gamma^2 - e\gamma + a = 0$ are

$$\gamma_{1,2} = \frac{e \pm \sqrt{e^2 - 4a}}{2}$$

The hypothesis gives $a \leq e - 1$, hence $e^2 - 4a \geq (e-2)^2$. Both γ_1 and γ_2 are then real. Assume $\gamma_1 \leq \gamma_2$. We obtain then $\gamma_1 \leq 1$ and $\gamma_2 \geq e - 1$.

The strict inequality in (3.2.1) shows that $\gamma(M)$ is outside the interval $[\gamma_1,\gamma_2]$. If $\gamma(M)<\gamma_1$, then $\gamma(M)<1$ and this contradicts Lemma 1.5. We conclude that $\gamma(M)>\gamma_2$, hence $\gamma(M)>e-1$. We recall that $\gamma(M)$ is an integer, cf. Lemma 1.5, and we conclude $\gamma(M)\geq e$. On the other hand, Lemma 1.6(1) implies $e>\gamma(M)$, a contradiction.

(2) We replace M with M_1, if necessary, so that we may assume $\mathfrak{m}^2M=0$. Proceed then as in (1), using Theorem 2.5. □

4. The Auslander-Reiten Conjecture

In this section we prove the conjecture of Auslander and Reiten (stated in the introduction) when the module is annihilated by $\mathfrak{m}^2$. More precise statements are obtained when $\mathfrak{m}^3=0$. We state our main results below. The proofs will follow later in the section.

4.1. **Theorem.** *Let $(R,\mathfrak{m},k)$ be an Artinian local ring with $\mathfrak{m}^3=0$ and M a finitely generated R-module.*

(1) *If $\operatorname{Ext}^i_R(M,M\oplus R)=0$ for four consecutive values of i with $i\geq 2$, then M is free.*
(2) *If R is Gorenstein and $\operatorname{Ext}^i_R(M,M)=0$ for some $i>0$, then M is free*

The second part was inspired by the following statement of Hoshino [12, 3.4]: If R is a finite dimensional self-injective local algebra (possibly non-commutative) over a field and the cube of its radical is zero, then any finitely generated R-module M with $\operatorname{Ext}^1_R(M,M)=0$ is free.

4.2. **Theorem.** *Let $(R,\mathfrak{m},k)$ be an Artinian local ring and M a finitely generated R-module with $\mathfrak{m}^2M=0$.*

If $\operatorname{Ext}^i_R(M,M\oplus R)=0$ for all i with $0<i\leq\max\{3,\nu(M),\nu(\mathfrak{m}M)\}$, then M is free.

4.3. *Remark.* Let $(R,\mathfrak{m},k)$ be an Artinian local ring with $\mathfrak{m}^2\neq 0$ and let M be a non-zero finitely-generated R-module with $\mathfrak{m}^2M=0$. If $\operatorname{Ext}^i_R(M,M)=0$ for some $i>0$, then $\gamma(M^\vee)=\gamma(M)^{-1}$.

Indeed, 2.7.1 gives $\operatorname{Tor}^R_i(M,M^\vee)=0$. Since $\mathfrak{m}^2$ is not zero, the modules $M^\vee$ and M are not free, hence k is not a direct summand in either of them, cf. Remark 1.1. Set $r(M)=\operatorname{rank}_k\operatorname{Soc}(M)$. As noted in Remark 2.3, $r(M)=\lambda(\mathfrak{m}M)$ and $r(M^\vee)=\lambda(\mathfrak{m}M^\vee)$. Using Matlis duality we obtain

$$\gamma(M^\vee)=\frac{\lambda(\mathfrak{m}M^\vee)}{\nu(M^\vee)}=\frac{r(M^\vee)}{r(M)}=\frac{\nu(M)}{\lambda(\mathfrak{m}M)}=\frac{1}{\gamma(M)}$$

4.4. **Proposition.** *Let $(R,\mathfrak{m},k)$ be an Artinian local ring. Let M be a non-zero finitely generated R-module such that $\mathfrak{m}^2M=0$. If any of the following conditions holds:*

(1) *$\operatorname{Ext}^i_R(M,M)=0$ for all i with $0<i\leq\max\{3,\nu(M),\nu(\mathfrak{m}M)\}$*
(2) *$\mathfrak{m}^3=0$ and $\operatorname{Ext}^i_R(M,M)=0$ for three consecutive values of $i>0$,*

then $\mathfrak{m}^2=0$, and M is either free or injective.

Proof. By 2.7.1 we have $\operatorname{Tor}_i(M,M^\vee)=0$ for all i as in the statement.

If $\mathfrak{m}^2=0$, then Remark 2.1 implies M or $M^\vee$ is free, hence M is free or injective.

From now in we assume $\mathfrak{m}^2\neq 0$. This implies, in particular, that neither M nor $M^\vee$ is free. By Remark 1.1, k is not a direct summand in M or $M^\vee$. Matlis duality and Remark 2.3 then yield $\nu(\mathfrak{m}M) = \nu(M^\vee)$, while Remark 4.3 gives $\gamma(M^\vee) = \gamma(M)^{-1}$. Lemma 1.5(1), respectively Theorem 2.5(1), give then $\gamma(M) \geq 1$ and $\gamma(M^\vee) \geq 1$, and we conclude $\gamma(M) = \gamma(M^\vee) = 1$.

We use the notation of the previous sections: $e = \nu(\mathfrak{m})$ and $a = \dim_k \operatorname{Soc}(R)$.

Assume that M satisfies (1). By Lemma 1.7 we have $2 - \gamma(M \otimes_R M^\vee) = e$, hence $e \leq 2$.

By Scheja [19, Satz 9] the ring R is then either a complete intersection, or a Golod ring. If it is a complete intersection, then $\operatorname{Ext}^2_R(M, M) = 0$ implies M is free, by Auslander, Ding, and Solberg [1, (1.8)], a contradiction. If it is Golod, but not a hypersurface, then $e = 2$ and $\gamma(M \otimes_R M^\vee) = 0$. Lemma 1.4(1) yields then $b_i(M) = \nu(M)$ and $b_i(M^\vee) = \nu(M^\vee)$ for $i = 1, 2, 3$. Since R is Golod, $P^R_k(t) = (1+t)(1-t-lt^2)^{-1}$ with $l \geq 1$, hence $b_3(k) = 2+3l \geq 5$. Since $M \otimes_R M^\vee$ is a sum of copies of k, we have then

$$\beta_3(M \otimes_R M^\vee) = b_3(k)\nu(M)\nu(M^\vee) \geq 5\nu(M)\nu(M^\vee)$$

On the other hand, Lemma 1.2 gives $\beta_3(M \otimes_R M^\vee) = 4\nu(M)\nu(M^\vee)$, and this leads to a contradiction.

Assume that M satisfies (2). Using 2.5(4) we obtain $a = 1$ and $e = \gamma(M) + \gamma(M^\vee) = 2$. Thus, R is Gorenstein and it follows that it is a complete intersection. Let j be an even integer among the three consecutive integers in the hypothesis. The hypothesis that $\operatorname{Ext}^j_R(M, M) = 0$ implies M is free, cf. Avramov and Buchweitz [5, 4.2], a contradiction. □

Proof of Theorem 4.2. Assume that M is not free. Proposition 4.4(1) then shows $\mathfrak{m}^2 = 0$ and M is injective. By 2.7.1, $\operatorname{Ext}^i_R(M, R) = 0$ implies $\operatorname{Tor}^R_i(M, \omega) = 0$ and Remark 2.1 shows that ω is free, hence R is Gorenstein. In this case, any finiteley generated injective R-module, and in particular M, is also free, a contradiction. □

4.5. Consider a short exact sequence of R-modules

$$0 \to N \to R^n \to M \to 0$$

Assume that $\operatorname{Ext}^{i+1}_R(M, R) = 0$ for some $i > 0$. A usual computation with the long exact sequences of Ext shows that $\operatorname{Ext}^i_R(M, M) = 0$ implies $\operatorname{Ext}^i_R(N, N) = 0$. If, in addition, $\operatorname{Ext}^i_R(M, R) = 0$ then $\operatorname{Ext}^i_R(M, M) = 0$ if and only if only if $\operatorname{Ext}^i_R(N, N) = 0$.

Proof of Theorem 4.1(1). The hypothesis and 4.5 imply that $\operatorname{Ext}^i_R(M_1, M_1)$ vanishes for three consecutive values of $i > 0$. Since $\mathfrak{m}^2 M_1 = 0$, we proceed as above, using Proposition 4.4(2). □

In order to prove part (2) of Theorem 4.1 we use the fact that for Gorenstein Artinian rings one can define negative Betti numbers and negative syzygies.

4.6. Let R be a Gorenstein Artinian local ring, F a free resolution of M and G a free resolution of M^*. Note that the complex G^* is acyclic, with $\mathrm{H}_0(G^*) = M^{**} \cong M$. Gluing together the complexes F and G^*, we obtain an exact complex P, which is called a *complete* resolution of M.

Furthermore, if M is a first syzygy in a minimal free resolution of some other module, and the resolutions F and G are chosen to be minimal, then the complex P is minimal. In this case, we have $\operatorname{rank}(P_i) = b_i(M)$ for all $i \geq 0$ and $M_i = \operatorname{Coker} \partial^P_i$. In general, the Betti numbers and syzygies of M are defined by setting

$b_i(M) = \operatorname{rank}(P_i)$ and $M_i = \operatorname{Coker} \partial_i^P$ for all integers i. Note that for any $j \geq i$ the module M_j is a $(j-i)$'th syzygy of M_i; in our notation: $M_j = (M_i)_{j-i}$.

Proof of Theorem 4.1(2). Assume that M is not free. Since R is Gorenstein, the hypothesis and 4.5 imply $\operatorname{Ext}_R^i(M_j, M_j) = 0$ for all j. Replacing M by M_1, we may assume $\mathfrak{m}^2 M = 0$. We can now use the notation of 4.6. The assumption that M is not free implies that both M and M^* have infinite projective dimension, hence M_j is not free for any j.

For all j we have thus $\mathfrak{m}^2 M_j = 0$. Since $\operatorname{Ext}_R^i(M_j, M_j) = 0$, 2.7.1 then gives $\operatorname{Tor}_i^R(M_j, M_j^*) = 0$. (Note that $\omega \cong R$, hence $M_j^* \cong M_j^\vee$). In particular, k is not a direct summand in any of the M_j's.

We set $e = \nu(\mathfrak{m})$ and $b_j = b_j(M)$ for all j. Since R is Gorenstein, $\operatorname{Soc}(R)$ is 1-dimensional. We use Lescot's results recalled in 2.2 to get:

$$b_j + b_{j-1} = e b_{j-2} \quad \text{and} \quad \nu(\mathfrak{m} M_j) = b_{j-1} \quad \text{for all} \quad j\,. \tag{4.1.1}$$

Recall that $\gamma(M) = \lambda(\mathfrak{m}M)/\nu(M)$. Using (4.1.1) and Remark 4.3 we have:

$$\gamma(M_j) = \frac{b_{j-1}}{b_j} \quad \text{and} \quad \gamma(M_j^*) = \frac{b_j}{b_{j-1}} \quad \text{for all} \quad j\,. \tag{4.1.2}$$

Since $\operatorname{Tor}_i^R(M_j, M_j^*) = 0$ for all j, Lemma 1.4 yields:

$$b_{i+j} = \big(\gamma(M_j^*) - \gamma(M_{j+i-1} \otimes_R M_j^*)\big) b_{j+i-1} \tag{4.1.3}$$

For all j we obtain:

$$b_{j+i} \leq \gamma(M_j^*) b_{j+i-1} = \frac{b_j}{b_{j-1}}\, b_{j+i-1}$$

where the inequality comes from (4.1.3) and the equality from (4.1.2). Equivalently:

$$\frac{b_{j+i}}{b_{j+i-1}} \leq \frac{b_j}{b_{j-1}} \quad \text{for all} \quad j\,.$$

Each $j = 0, 1, \cdots, i-1$ yields thus a non-increasing sequence $(b_{j+ni}/b_{j+ni-1})_n$. Let L_j denote the limit of the j'th sequence.

If $L_j < 1$ for some $j = 0, 1, \ldots, i-1$ it follows that there exists an eventually strictly decreasing subsequence of $\{b_n\}$. This implies that $b_n = 0$ for some $n \gg 0$, a contradiction.

Thus, $L_j \geq 1$ for all j. This implies $b_n \leq b_{n+1}$ for all n. If $b_{n_0} < b_{n_0+1}$ for some n_0, then we obtain

$$1 < \frac{b_{n_0+1}}{b_{n_0}} \leq \frac{b_{n_0-i+1}}{b_{n_0-i}} \leq \cdots$$

hence

$$b_{n_0+1} > b_{n_0} \geq b_{n_0-i+1} > b_{n_0-i} \geq b_{n_0-2i+1} > b_{n_0-2i} \geq \cdots$$

It follows that $b_n = 0$ for some $n \ll 0$, a contradiction. In conclusion, $b_n = b_{n+1}$ for all n. We use then (4.1.1) to obtain $e = 2$. It follows that R is a complete intersection. Since the Betti numbers of any M_j are constant, by a result of Eisenbud [9, 4.1] then $M_{j+n} \cong M_{j+n+2}$ for all $n > 0$. We conclude $M_j \cong M_{j+2}$ for all j.

If i is even, then M has finite projective dimension by [5, 4.2], hence it is free, a contradiction.

If i is odd, then $M \cong M_{-i+1}$, hence

$$\operatorname{Ext}_R^1(M, M) \cong \operatorname{Ext}_R^1(M, M_{-i+1}) \cong \operatorname{Ext}_R^i(M, M) = 0$$

Since the Betti numbers of M are constant, (4.1.2) gives $\gamma(M^*) = 1$. Taking $i = 1$ and $j = 0$ in (4.1.3) we get $b_1 = \big(1 - \gamma(M \otimes_R M^*)\big)b_0$, and it follows $\gamma(M \otimes_R M^*) = 0$. This means that $\mathfrak{m}(M \otimes_R M^*) = 0$. However, $M \otimes_R M^*$ is the Matlis dual of $\operatorname{Hom}_R(M, M)$ and the later is annihilated by $\mathfrak{m}$ only when $\mathfrak{m}M = 0$. In view of the hypothesis, this implies that M is free, which provides the desired contradiction. $\square$

5. Vanishing of Tor and Loewy length

In this section, unless otherwise stated, we assume that $(R, \mathfrak{m}, k)$ is an Artinian local ring. We recall that the Loewy length of R, denoted $\ell\ell(R)$, is the smallest integer n with $\mathfrak{m}^n = 0$.

We propose the following conjecture:

5.1. **Conjecture.** *Assume that M, N are nonzero modules with $\mathfrak{m}^2M = \mathfrak{m}^2N = 0$. If $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > 0$, then $\mathfrak{m}^3 = 0$.*

5.2. One can ask a more general question: Assume that M, N are nonzero modules with $\mathfrak{m}^pM = \mathfrak{m}^qN = 0$. If $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > 0$, does it follow that $\mathfrak{m}^{p+q-1} = 0$? This is trivially true when $p = 1$ or $q = 1$, or when one of p, q is greater than $\ell\ell(R)$. The conjecture takes up the case $p = 2 = q$.

5.3. In view of Lemma 1.8, the conjecture holds whenever $\mathfrak{m}^3 = 0$ or $P_k^R(t) \neq (1 - at)(1 - bt)^{-1}(1 - ct)^{-1}$ with rational numbers $a \geq 0$ and $b, c > 0$. The class of such rings include: complete intersection rings of codimension different from 2, generalized Golod rings, Koszul rings, rings with irrational Poincaré series and many others. More evidence for the conjecture can also be gathered from the preceding two sections.

Theorem 5.4 below establishes the conjecture in yet another important case.

We say that the local ring R is standard graded if it has a decomposition $R = R_0 \oplus R_1 \oplus \cdots \oplus R_h$ such that $R_iR_j \subseteq R_{i+j}$ for all $i, j \in [0, h]$, $R_0 = k$ and $R = R_0[R_1]$ (it is thus generated in degree one).

5.4. **Theorem.** *Let $(R, \mathfrak{m})$ be a standard graded local ring, and let M, N be nonzero finitely generated R-modules satisfying $\mathfrak{m}^2M = \mathfrak{m}^2N = 0$. If $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > 0$, then $\mathfrak{m}^3 = 0$.*

Let L, U be any two R-modules. We set $\overline{L} = L/\mathfrak{m}L$ and we consider the short exact sequence

$$0 \longrightarrow \mathfrak{m}L \xrightarrow{\mu_L} L \longrightarrow \overline{L} \longrightarrow 0$$

and the induced long exact sequence:

$$\cdots \to \operatorname{Tor}_{i+1}^R(U, \overline{L}) \xrightarrow{\Delta_i(U,L)} \operatorname{Tor}_i^R(U, \mathfrak{m}L) \xrightarrow{\operatorname{Tor}_i^R(U,\mu_L)} \operatorname{Tor}_i^R(U, L) \to \dots$$

where $\Delta_i(U, L)$ denote the connecting homomorphisms.

5.5. **Lemma.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and let M be a non-zero finitely generated R-module with $\mathfrak{m}^2M = 0$. Let $j \geq 2$ be an integer. If $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \in [1, j]$, then $\operatorname{Tor}_i^R(k, \mu_N) = 0$ for all $i \in [0, j - 1]$.*

Proof. (1) We will show, equivalently, that $\Delta_i(k,N)$ is surjective for all $i \in [0, j-1]$. Of course, this is true when $i = 0$. We prove the claim by induction. Assume that it is true for $i = n$ and we prove it for $i = n+1$, with $0 \le n < j-1$.

In the diagram below the horizontal lines are long exact sequences of the type considered above.

$$\begin{tikzcd}
\operatorname{Tor}^R_{n+2}(M,\overline{N}) \arrow[r] \arrow[d, "\Delta_{n+1}(M,N)"', "\cong"] & \operatorname{Tor}^R_{n+2}(\overline{M},\overline{N}) \arrow[r] \arrow[d, "\Delta_{n+1}(\overline{M},N)"'] & \operatorname{Tor}^R_{n+1}(\mathfrak{m}M,\overline{N}) \arrow[r] \arrow[d, two heads, "\Delta_n(\mathfrak{m}M,N)"'] & \operatorname{Tor}^R_{n+1}(M,\overline{N}) \arrow[d, hook, "\Delta_n(M,N)"'] \\
\operatorname{Tor}^R_{n+1}(M,\mathfrak{m}N) \arrow[r] & \operatorname{Tor}^R_{n+1}(\overline{M},\mathfrak{m}N) \arrow[r] & \operatorname{Tor}^R_{n}(\mathfrak{m}M,\mathfrak{m}N) \arrow[r] & \operatorname{Tor}^R_{n}(M,\mathfrak{m}N)
\end{tikzcd}$$

By [8, V, 4.1] the exterior squares commutes and the interior one anticommutes. The map $\Delta_{n+1}(M,N)$ is bijective because $\operatorname{Tor}^R_{n+2}(M,N) = \operatorname{Tor}^R_{n+1}(M,N) = 0$. Also, the map $\Delta_n(\mathfrak{m}M,N)$ is surjective by the induction hypothesis, using the fact that $\mathfrak{m}M$ is a finite direct sum of copies of k, and $\Delta_n(M,N)$ is injective because $\operatorname{Tor}^R_{n+1}(M,N) = 0$. By the "Five Lemma" we conclude that the map $\Delta_{n+1}(\overline{M},N)$ is surjective. Since $\overline{M}$ is a finite direct sum of copies of k, we obtain that $\Delta_{n+1}(k,N)$ is surjective, and this finishes the induction argument. $\square$

Proof of Theorem 5.4. We may assume that M and N are not free. By Lemma 5.5 we have $\operatorname{Tor}^R_i(k,\mu_{N_1}) = 0$ for all $i \ge 0$. Choose such an i and set $F = R^{b_0(N)}$. By Lemma 1.6(2) we have $\mathfrak{m}N_1 = \mathfrak{m}^2F$. Consider the following commutative diagram, in which the vertical maps are induced by the inclusion $N_1 \hookrightarrow \mathfrak{m}F$:

$$\begin{tikzcd}
\operatorname{Tor}^R_i(k,\mathfrak{m}N_1) \arrow[r, "\operatorname{Tor}^R_i(k,\mu_{N_1})=0"] \arrow[d, "\cong"] & \operatorname{Tor}^R_i(k,N_1) \arrow[d] \\
\operatorname{Tor}^R_i(k,\mathfrak{m}^2F) \arrow[r, "\operatorname{Tor}^R_i(k,\mu_{\mathfrak{m}F})"] & \operatorname{Tor}^R_i(k,\mathfrak{m}F)
\end{tikzcd}$$

We conclude that $\operatorname{Tor}^R_i(k,\mu_{\mathfrak{m}F}) = 0$, hence $\operatorname{Tor}^R_i(k,\mu_{\mathfrak{m}}) = 0$ for all i. Equivalently, the map $\operatorname{Ext}^i_R(k,k) \to \operatorname{Ext}^i_R(R/\mathfrak{m}^2,k)$ induced by the projection $R/\mathfrak{m}^2 \to k$ is zero for all i, hence [18, Corollary 1] implies that the algebra $\operatorname{Ext}^*_R(k,k)$ (with Yoneda product) is generated by its elements of degree 1. This means that the k-algebra R is Koszul, hence $P^R_k(t) = \operatorname{Hilb}_R(-t)^{-1}$, cf. [16, Theorem 1.2]. Comparing with the relation of Remark 1.8 we conclude that $\operatorname{Hilb}_R(t)$ is a polynomial of degree at most 2, hence $\mathfrak{m}^3 = 0$. $\square$

6. Rings of type at most 2

In this section we show that the conjecture of Tachikawa stated in the introduction holds for Cohen-Macaulay rings of type at most 2.

6.1. **Theorem.** *Let R be a Cohen-Macaulay local ring with a canonical module ω and such that* $\operatorname{type}(R) \le 2$.

(1) *If* $\operatorname{Tor}^R_2(\omega,\omega) = 0$*, then R is Gorenstein.*
(2) *If* $\operatorname{Ext}^i_R(\omega,R) = 0$ *for $i = 1, 2$, then R is Gorenstein.*

The proof will be given at the end of the section, after discussing some preliminaries. We recall a well-known fact:

6.2**.** Let R be a commutative local ring and consider a short exact sequence of R-modules:

$$0 \longrightarrow N \xrightarrow{\varphi} R^n \xrightarrow{\psi} M \longrightarrow 0$$

The map φ induces a natural map $\wedge_R^n(\varphi)\colon \wedge_R^n(N) \to \wedge_R^n(R^n)$. If $a \in R$ is in the image of this map, via the identification of R with $\wedge_R^n(R^n)$, then $aM = 0$.

In particular, if M is faithful, then $\wedge_R^n(\varphi) = 0$.

6.3**. Proposition.** *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M a finitely generated R-module. Consider a short exact sequence of R-modules*

$$0 \longrightarrow N \xrightarrow{\varphi} R^2 \xrightarrow{\psi} M \longrightarrow 0$$

If $\operatorname{Tor}_2^R(M, M) = 0$ and M is faithful, then $\nu(N) \leq 1$.

Proof. For any module L we define a map $\iota^L\colon \wedge_R^2(L) \to L \otimes_R L$ given by

$$\iota^L(x \wedge y) = x \otimes y - y \otimes x$$

The hypothesis implies $\operatorname{Tor}_1^R(M, N) = 0$. It follows that the induced map

$$\varphi \otimes_R N\colon N \otimes_R N \to R^2 \otimes_R N$$

is injective. The map $\varphi \otimes_R \varphi\colon N \otimes_R N \to R^2 \otimes_R R^2$ is the composition

$$N \otimes_R N \xrightarrow{\varphi \otimes_R N} R^2 \otimes_R N \xrightarrow{R^2 \otimes_R \varphi} R^2 \otimes_R R^2$$

Both maps are injective, hence so is $\varphi \otimes_R \varphi$.

Recall from 6.2 that $\wedge_R^2(\varphi) = 0$. The commutative diagram

$$\begin{tikzcd}
\wedge_R^2(N) \arrow[r, "\iota^N"] \arrow[d, "\wedge_R^2(\varphi)"'] & N \otimes_R N \arrow[d, hook, "\varphi \otimes_R \varphi"] \\
\wedge_R^2(R^2) \arrow[r, "\iota^R"] & R^2 \otimes_R R^2
\end{tikzcd}$$

then yields $\iota^N = 0$. In particular, the map $\iota^N \otimes_R k$ is zero. Note that this map can be identified with $\iota^{N \otimes_R k}$. As $N \otimes_R k$ is a finite direct sum of copies of k, the map $\iota^{N \otimes_R k}$ is clearly injective. It follows that $\wedge_R^2(N \otimes_R k) = 0$, hence $N \otimes_R k \cong k$. Nakayama's Lemma then shows that N is cyclic. $\square$

Proof of Theorem 6.1. (1) Assume that R is not Gorenstein, hence $\operatorname{type}(R) = 2$. Set $N = \omega_1$. We denote $e(L)$ the multiplicity of an R-module L. Since $e(\omega) = e(R)$, we use the fact that multiplicity is additive on short exact sequences of maximal Cohen-Macaulay modules to obtain $e(N) = e(R)$. By Proposition 6.3, N is cyclic, hence there is a surjection $R \to N$. If K is the kernel of this map, then $e(K) = 0$, hence N is free, a contradiction.

(2) By [6, B.4] we have $\operatorname{Tor}_i^R(\omega, \omega) = 0$ for $i = 1, 2$, so we can apply (1). $\square$

Acknowledgements

The authors want to thank Luchezar Avramov, Ragnar Buchweitz, Graham Leuschke, and Roger Wiegand for their comments and suggestions.

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