CONDITIONS FOR THE YONEDA ALGEBRA
OF A LOCAL RING
TO BE GENERATED IN LOW DEGREES

JUSTIN HOFFMEIER AND LIANA M. ŞEGA

Abstract. The powers $m^n$ of the maximal ideal $m$ of a local Noetherian ring $R$ are known to satisfy certain homological properties for large values of $n$. For example, the homomorphism $R \to R/m^n$ is Golod for $n \gg 0$. We study when such properties hold for small values of $n$, and we make connections with the structure of the Yoneda Ext algebra, and more precisely with the property that the Yoneda algebra of $R$ is generated in degrees 1 and 2. A complete treatment of these properties is pursued in the case of compressed Gorenstein local rings.

Introduction

Let $(R, m, k)$ be a local ring, that is, a commutative noetherian ring $R$ with unique maximal ideal $m$ and $k = R/m$. For $n \geq 1$ we let $\nu_n : m^n \to m^{n-1}$ denote the canonical inclusion and for each $i \geq 0$ we consider the induced maps $\text{Tor}^R_i(\nu_n, k) : \text{Tor}^R_i(m^n, k) \to \text{Tor}^R_i(m^{n-1}, k)$.

Using the terminology of [2], we say that $m^n$ is a small submodule of $m^{n-1}$ if $\text{Tor}^R_i(\nu_n, k) = 0$ for all $i \geq 0$. This condition implies that the canonical projection $\rho_n : R \to R/m^n$ is a Golod homomorphism, but the converse may not hold.

Levin [9] showed that $m^n$ is a small submodule of $m^{n-1}$ for all sufficiently large values of $n$. On the other hand, the fact that $m^n$ is a small submodule of $m^{n-1}$ for small values of $n$ is an indicator of strong homological properties. It is known that $m^2$ is a small submodule of $m$ if and only if the Yoneda algebra $\text{Ext}^*_R(k, k)$ is generated in degree 1, cf. [12, Corollary 1]. More generally, we show:

Theorem 1. Let $(R, m, k)$ be a local ring. Let $\hat{R} = Q/I$ be a minimal Cohen presentation of $R$, with $(Q, n, k)$ a regular local ring and $I \subseteq n^t$. Let $t$ be an integer such that $I \subseteq n^t$. The following statements are then equivalent:

1. $m^t$ is a small submodule of $m^{t-1}$;
2. $\rho_t : R \to R/m^t$ is Golod;
3. $\rho_n : R \to R/m^n$ is Golod for all $n$ such that $t \leq n \leq 2t - 2$;
4. $I \cap n^{t+1} \subseteq n^t$ and the algebra $\text{Ext}^*_R(k, k)$ is generated by $\text{Ext}^1_R(k, k)$ and $\text{Ext}^2_R(k, k)$.

If $R$ is artinian, its socle degree is the largest integer $s$ with $m^s \neq 0$. When $R$ is a compressed Gorenstein local ring (see Section 3 for a definition) of socle degree $s \neq 3$, we determine all values of the integer $n$ for which the homomorphism

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\( \rho_n \) is Golod, respectively for which \( m^n \) is a small submodule of \( m^{n-1} \), and we use Theorem 1 to establish part (3) below.

**Theorem 2.** Let \((R, m, k)\) be a compressed Gorenstein local ring of socle degree \( s \).

Assume \( 2 \leq s \neq 3 \) and let \( t \) denote the smallest integer such that \( 2t \geq s + 1 \). If \( n \geq 1 \), then the following hold:

1. \( m^n \) is a small submodule of \( m^{n-1} \) if and only if \( n > s \) or \( n = s + 2 - t \).
2. \( \rho_n : R \rightarrow R/m^n \) is Golod if and only if \( n \geq s + 2 - t \).
3. If \( s \) is even, then \( \text{Ext}_{R}(k, k) \) is generated by \( \text{Ext}^{2}_{R}(k, k) \) and \( \text{Ext}^{2}_{R}(k, k) \).

The conclusion of (3) does not hold when \( s \) is odd, see Corollary 3.7.

Section 1 provides definitions and properties of the homological notions of interest. Theorem 1 is proved in Section 2 and Theorem 2 is proved in Section 3.

1. Preliminaries

Throughout the paper \((R, m, k)\) denotes a commutative noetherian local ring with maximal ideal \( m \) and residue field \( k \). Let \( M \) be a finitely generated \( R \)-module.

We denote by \( \hat{R} \) the completion of \( R \) with respect to \( m \). A minimal Cohen presentation of \( R \) is a presentation \( \hat{R} = Q/I \), with \( Q \) a regular local ring with maximal ideal \( n \) and \( I \) an ideal with \( I \subseteq n^2 \). We know that such a presentation exists, by the Cohen structure theorem.

We denote by \( R^{\#} \) the associated graded ring with respect to \( m \), and by \( M^{\#} \) the associated graded module with respect to \( m \). We denote by \( (R^{\#})_j \) the \( j \)-th graded component of \( R^{\#} \). For any \( x \in R \) we denote by \( x^{\#} \) the image of \( x \) in \( m^{j}/m^{j+1} = (R^{\#})_j \), where \( j \) is such that \( x \in m^j \setminus m^{j+1} \). For an ideal \( J \) of \( R \), we denote by \( J^{\#} \) the homogeneous ideal generated by the elements \( x^{\#} \) with \( x \in J \).

**Remark 1.1.** With \( \hat{R} = Q/I \) as above, the following then hold:

1. \( I \subseteq n^t \) if and only if \( \text{rank}_m(m^{t-1}/m^t) = \left( \frac{e + t - 2}{e - 1} \right) \), where \( e \) denotes the minimal number of generators of \( m \).
2. Assume \( t \geq 2 \) and \( I \subseteq n^t \). Then \( I \cap n^{t+1} \subseteq nI \) if and only if the map
   \[ \text{Ext}^{2}_{\rho_t}(k, k) : \text{Ext}^{2}_{R/m^t}(k, k) \rightarrow \text{Ext}^{2}_{R}(k, k) \]
   induced by the canonical projection \( \rho_t : R \rightarrow R/m^t \) is surjective.

To prove (1), note that \( (\hat{R})_{e}^{\#} = Q^{\#}/I^{\#} \) and \( Q^{\#} \) is isomorphic to a polynomial ring over \( k \) in \( e \) variables of degree 1. We have that \( I \subseteq n^t \) if and only if \( I^{\#} \subseteq (m^{\#})^t \), which is equivalent to \( (Q^{\#})_{t-1} = (Q^{\#}/I^{\#})_{t-1} \) and thus to

\[ \text{rank}_m(Q^{\#})_{t-1} = \text{rank}_m((\hat{R})_{t-1}^{\#}) \, . \]

Therefore, (1) follows by noting that \( \text{rank}_m(Q^{\#})_{j} = \left( \frac{e - 1 + j}{e - 1} \right) \) and \( \text{rank}_m((\hat{R})_{j}^{\#}) = \text{rank}_m(m^t/m^{t+1}) \) for each \( j \).

For a proof of (2), see Şega [15, 4.3], noting that the map \( \text{Ext}^{2}_{\rho_t}(k, k) \) is surjective if and only if the induced map

\[ \text{Tor}^{2}_{\rho_t}(k, k) : \text{Tor}^{2}_{R}(k, k) \rightarrow \text{Tor}^{2}_{R/m^t}(k, k) \]

is injective.
Definition. Let $\hat{R} = Q/I$ be a minimal Cohen presentation of $R$. Let $t \geq 2$ be an integer. We say that the local ring $R$ is $t$-homogeneous if $I \subseteq n^t$ and $I \cap n^{t+1} \subseteq nI$. Remark 1.1 shows that this definition does not depend on the choice of the minimal Cohen presentation.

We set

$$v(R) = \sup \{ t \geq 0 \mid I \subseteq n^t \}.$$  

Note that, if $R$ is $t$-homogeneous and $I \neq 0$, then $t = v(R)$.

Remark 1.2. The terminology of two-homogeneous algebra was previously used by Lőfwall [11], with a different meaning, in the context of augmented graded algebras.

Lemma 1.3. Let $\hat{R} = Q/I$ be a minimal Cohen presentation of $R$. If the ideal $I^*$ of the polynomial ring $Q^k$ is generated by homogeneous polynomials of degree $t$, then the ring $R$ is $t$-homogeneous.

Proof. Assume $I^*$ is generated by homogeneous polynomials of degree $t$. In particular, it follows that $I \subseteq n^t$. To prove $I \cap n^{t+1} \subseteq nI$, we will show $I \cap n^{t+1} \subseteq nI + n^t$ for all $j \geq 0$. The Krull intersection theorem then gives the conclusion.

Let $x \in I \cap n^{t+1}$ and let $a \geq 1$ be the smallest integer such that $x \notin n^{t+1+a}$. Then $x^*$ is an element of degree $t + a$ of $I^*$. Since $I^*$ is generated by homogeneous elements of degree $t$, we can write

$$x^* = \sum y_i^* z_i^*$$

with $y_i^* \in (Q^k)_a$ and $z_i^* \in I^* \cap (Q^k)_t$ for each $i$, where $y_i \in n^a$ and $z_i \in I \cap n^t$. Set

$$x_1 = x - \sum y_i z_i,$$

and note that $x_1 \in I \cap n^{t+a+1}$ and $x - x_1 \in nI$. In particular $x \in nI + n^{t+a+1}$. Applying the argument above to $x_1$, we obtain an element $a_1$ such that $a_1 > a$, and an element $x_2$ such that $x_2 \in I \cap n^{t+a_1+1}$ and $x_1 - x_2 \in nI$. In particular $x_1$, and thus $x$, are elements of $nI + n^{t+a_1+1}$. An inductive argument produces a sequence of integers $1 < a_1 < a_2 < \ldots$ such that $x \in nI + n^{t+a_i+1}$ for all $i$, and gives the desired conclusion. 

Remark 1.4. The converse of the lemma does not hold. This can be seen by considering the 2-homogeneous local ring $R = k[[x,y]]/(x^2 + y^3, xy)$, for which $R^k = k[x,y]/(x^2, y^4, xy)$.

We now proceed to provide definitions for the homological notions of interest, and recall some of their properties.

The Poincaré series $P^R_{M}(z)$ of $M$ is the formal power series

$$P^R_{M}(z) = \sum_{i \geq 0} \text{rank}_k(\text{Tor}^R_i(M, k))z^i.$$ 

1.5. Golod rings, modules, and homomorphisms. Let $(S, s, k)$ be a local ring and $\varphi: R \rightarrow S$ be a surjective homomorphism of local rings. Following Levin [10], we say that an $S$-module $M$ is a $\varphi$-Golod if the following equality is satisfied:

$$P^S_{M}(z) = \frac{P^R_{M}(z)}{(1 - z(P^R_{S}(z)) - 1))}. $$

We say that $\varphi$ is a Golod homomorphism if $k$ is a $\varphi$-Golod module.
The ring $R$ is said to be a Golod ring if the canonical projection $Q \to \widehat{R}$ is a Golod homomorphism, where $\widehat{R} = Q/I$ is a minimal Cohen presentation. This definition is independent of the choice of representation by [1, Lemma 4.1.3]. Note that this definition is equivalent to the definition given by Avramov in terms of Koszul homology in [1, §5]. A classical example of a Golod ring is $Q/n^j$ for any $j \geq 2$, where $(Q, n, k)$ is a regular local ring, as first observed by Golod [5].

1.6. Small homomorphisms. Let $\varphi : R \to S$ be a surjective ring homomorphism as above and consider the induced maps

$$\text{Ext}^i_{\varphi}(k, k) : \text{Ext}^i_S(k, k) \to \text{Ext}^i_R(k, k)$$

and

$$\text{Tor}^i_{\varphi}(k, k) : \text{Tor}^i_R(k, k) \to \text{Tor}^i_S(k, k).$$

We say that $\varphi$ is small if $\text{Ext}^*_\varphi(k, k)$ is surjective or, equivalently, if $\text{Tor}^*_\varphi(k, k)$ is injective. Note that $\text{Tor}^1_{\varphi}(k, k)$ can be identified with the canonical map $m/m^2 \to s/s^2$ induced by $\varphi$. Thus $\text{Tor}^1_{\varphi}(k, k)$ is an isomorphism if and only if $\text{Ker}(\varphi) \subseteq m^2$.

For convenience we collect below a few known facts about small homomorphisms:

1. If $\varphi$ is Golod then $\varphi$ is small, see Avramov [2, 3.5].
2. If $S$ is a Golod ring and $\varphi$ is small, then $\varphi$ is Golod, see Šega [14, 6.7].

1.7. Inert modules. Let $\kappa : P \to R$ be a surjective homomorphism of local rings. Following Lescot [7], we say that an $R$-module $M$ is inert by $\kappa$ if the following equality holds:

$$P_R k_m(z) P_P k_m(z) = P_P k_m(z) P_R M(z).$$

If $\kappa$ is a Golod homomorphism the following are equivalent:

1. $M$ is $\kappa$-Golod;
2. $\text{Tor}^i_{\kappa}(M, k)$ is injective for all $i$;
3. $M$ is inert by $\kappa$.

The equivalence of (1) and (3) is a direct consequence of the definitions and the equivalence of (1) with (2) is given by Levin [10, 1.1].

1.7.1. Consider a sequence of surjective homomorphisms of local rings

$$R \xrightarrow{\alpha} S \xrightarrow{\beta} T.$$  

Lescot [7, Theorem 3.6] shows that a $T$-module $M$ is inert by $\beta \circ \alpha$ if an only if $M$ is inert by $\beta$ and $M$ is inert by $\alpha$, when considered as an $S$-module.

1.7.2. Let $Q$ be a regular local ring with maximal ideal $n$ and an ideal $J$ such that $J \subseteq n^t$ with $t \geq 2$. Set $S = Q/J$ and $\overline{n} = n/J$. If a finitely generated $S$-module $N$ is annihilated by $\overline{n}^{t-1}$ then $N$ is inert by the natural projection $\kappa : Q \to S$, see [7, 3.7 and 3.11].

We end this section with introducing some more notation.

1.8. If $(R, m, k)$ is a local ring and $M$ is a finite $R$-module, we let $\nu_M : mM \to M$ denote the canonical inclusion and consider the induced maps

$$\text{Tor}^i_R(\nu_M, k) : \text{Tor}^i_R(mM, k) \to \text{Tor}^i_R(M, k).$$

They fit into the long exact sequence

$$\cdots \to \text{Tor}^i_R(mM, k) \xrightarrow{\text{Tor}^i_R(\nu_M, k)} \text{Tor}^i_R(M, k) \to \text{Tor}^i_R(M/mM, k) \to \cdots.$$
If \( \Lambda \) is a graded vector space over \( k \), we set \( H_\Lambda(z) = \sum_{i \geq 0} \) rank\(_i(\Lambda)\)z\(^i\); this formal power series is called the Hilbert series of \( \Lambda \). Since rank is additive on exact sequences, a rank count in the exact sequence above gives

\[
P_M^R(z) - aP_M^R(z) + zP_M^R = (1+z)H_{\text{Im}(\text{Tor}_{P(\nu_M,k)})}(z)
\]

where \( a = \text{rank}_k(M/mM) \). We set

\[
T_M^R(z) = H_{\text{Im}(\text{Tor}_{P(\nu_M,k)})}(z).
\]

2. Homological properties of powers of the maximal ideal

The purpose of this section is to prove Theorem 1 in the introduction, restated as Theorem 2.5 below.

For each integer \( j \) let

\[
\rho_j : R \to R/m^j \quad \text{and} \quad \nu_j : m^j \to m^{j-1}
\]
denote the canonical projection, respectively the inclusion, and consider the induced maps

\[
\text{Tor}_i^P(R/m^{j-1}, k) : \text{Tor}_i^R(R/m^{j-1}, k) \to \text{Tor}_i^R(R/m^{j-1}, k)
\]

\[
\text{Tor}_i^R(\nu_j, k) : \text{Tor}_i^R(m^j, k) \to \text{Tor}_i^R(m^{j-1}, k).
\]

Using the terminology in the appendix of [2], we say that \( m^j \) is a small submodule of \( m^{j-1} \) if \( \text{Tor}_i^R(\nu_j, k) = 0 \) for all \( i \geq 0 \).

Remark 2.1. For \( i, j \geq 0 \) let

\[
\eta_i^j : \text{Tor}_i^R(R/m^j, k) \to \text{Tor}_i^R(R/m^{j-1}, k)
\]
denote the map induced by the canonical projection \( R/m^j \to R/m^{j-1} \).

Note that \( \text{Tor}_i^R(\nu_j, k) = 0 \) if and only if \( \eta_i^{j+1} = 0 \). Indeed, this is a standard argument, using the canonical isomorphisms \( \text{Tor}_i^R(R/m^j, k) \cong \text{Tor}_i^R(m^j, k) \) which arise as connecting homomorphisms in the long exact sequence associated to the exact sequence

\[
0 \to m^n \to R \to R/m^n \to 0,
\]

with \( n = j \) and \( n = j - 1 \).

2.2. We state here a needed result of Rossi and Sèga [13, Lemma 1.2]:

Let \( \pi : (P, p, k) \to (R, m, k) \) be a surjective homomorphism of local rings. Assume there exists an integer \( a \) such that

1. The map \( \text{Tor}_i^P(R, k) \to \text{Tor}_i^P(R/m^a, k) \) induced by the natural projection \( R \to R/m^a \) is zero for all \( i > 0 \).

2. The map \( \text{Tor}_i^P(m^{2a}, k) \to \text{Tor}_i^P(m^a, k) \) induced by the inclusion \( m^{2a} \hookrightarrow m^a \) is zero for all \( i \geq 0 \).

Then \( \pi \) is a Golod homomorphism.

Proposition 2.3. Let \( (R, m, k) \) be a local ring and let \( j \geq 2 \) be an integer. The following are equivalent:

1. \( m^j \) is a small submodule of \( m^{j-1} \);

2. \( \text{Tor}_i^P(R/m^{j-1}, k) \) is injective for all \( i \geq 0 \);

3. \( \rho_j \) is Golod and \( R/m^{j-1} \) is inert by \( \rho_j \).

If these conditions hold, then \( \rho_j \) is Golod for all integers \( l \) with \( j \leq l \leq 2j - 2 \).
Proof. (1)⇒(2): Let \( i \geq 0 \). Set \( \mathfrak{m}^{i-1} = \mathfrak{m}^i / \mathfrak{m}^j \). Consider long exact sequences associated to the exact sequence

\[
0 \to \mathfrak{m}^{i-1} \to R / \mathfrak{m}^i \to R / \mathfrak{m}^{i-1} \to 0
\]

and create the following commutative diagram with exact columns.

\[
\begin{array}{ccc}
\text{Tor}^R_{i+1}(R / \mathfrak{m}^{i-1}, k) & \longrightarrow & \text{Tor}^R_{i+1}(R / \mathfrak{m}^{i}, k) = 0 \\
\eta_{i+1} & \downarrow & \\
\text{Tor}^R_{i+1}(R / \mathfrak{m}^{i-1}, k) & \longrightarrow & \text{Tor}^R_{i+1}(R / \mathfrak{m}^{i-1}, k) \\
\Delta_i & \downarrow & \\
\text{Tor}^R(\mathfrak{m}^{i-1}, k) & \longrightarrow & \text{Tor}^R(\mathfrak{m}^{i-1}, k)
\end{array}
\]

By Remark 2.1, the hypothesis that \( \text{Tor}^R_i(\nu_j, k) = 0 \) implies that \( \eta_{i+1} = 0 \), and thus the connecting homomorphism \( \Delta_i \) is injective.

Levin’s proof of [9, 3.15] shows \( \text{Tor}^R_i(\nu_j, k) = 0 \) for all \( i \) implies \( \rho_j \) is Golod. (This also follows from the last part of the proof.) In particular, the map \( \rho_j \) is small by 1.6(1). Since \( \mathfrak{m}^{i-1} \) is a direct sum of copies of \( k \), it follows that \( \text{Tor}^R_i(\mathfrak{m}^{i-1}, k) \) is injective.

The bottom commutative square yields that \( \text{Tor}^R_{i+1}(R / \mathfrak{m}^{i-1}, k) \) is injective.

(2)⇒(1): Assuming that \( \text{Tor}^R_{i+1}(R / \mathfrak{m}^{i-1}, k) \) is injective, the top square in the commutative diagram above gives that \( \eta_{i+1} = 0 \), and thus \( \text{Tor}^R_i(\nu_j, k) = 0 \) by Remark 2.1.

(1)⇒(3): As mentioned above, \( \text{Tor}^R_i(\nu_j, k) = 0 \) for all \( i \) implies that \( \rho_j \) is Golod. Since we already proved (1)⇒(2), we know that \( \text{Tor}^R_i(R / \mathfrak{m}^{i-1}, k) \) is injective for all \( i \). By 1.7, \( R / \mathfrak{m}^{i-1} \) is then inert by \( \rho_j \).

(3)⇒(2): see 1.7.

Fix now \( l \) such that \( j \leq l \leq 2j - 2 \). We prove the last assertion of the proposition by applying 2.2, with \( \alpha = \rho_l \) and \( a = j - 1 \). Set \( \overline{R} = R / \mathfrak{m}^l \) and \( \overline{\mathfrak{m}} = \mathfrak{m} / \mathfrak{m}^l \). Let

\[
\overline{p}_{j-1} : \overline{R} \to \overline{R} / \overline{\mathfrak{m}}^{i-1}
\]

denote the canonical projection. To satisfy the first hypothesis of 2.2, we will show that the induced map

\[
\text{Tor}^R_i(\overline{p}_{j-1}, k) : \text{Tor}^R_i(\overline{R}, k) \to \text{Tor}^R_i(\overline{R} / \overline{\mathfrak{m}}^{i-1}, k)
\]

is zero for all \( i > 0 \). Since \( l \geq j \), we have \( \overline{R} / \overline{\mathfrak{m}}^{i-1} = R / \mathfrak{m}^{i-1} \) and \( \text{Tor}^R_i(\overline{p}_{j-1}, k) \) factors through

\[
\eta'_i : \text{Tor}^R_i(R / \mathfrak{m}^{j}, k) \to \text{Tor}^R_i(R / \mathfrak{m}^{i-1}, k).
\]

Since \( \text{Tor}^R_i(\nu_j, k) = 0 \) for all \( i \geq 0 \) by assumption, we have that \( \eta'_i = 0 \) for all \( i > 0 \) by Remark 2.1. Hence \( \text{Tor}^R_i(\overline{p}_{j-1}, k) = 0 \) for all \( i > 0 \).

To satisfy the second hypothesis of 2.2, we need to show that the induced map

\[
\text{Tor}^R_i(\overline{\mathfrak{m}}^{(j-1)}, k) \to \text{Tor}^R_i(\overline{\mathfrak{m}}^{i-1}, k)
\]

induced by the inclusion \( \overline{\mathfrak{m}}^{(j-1)} \to \overline{\mathfrak{m}}^{i-1} \) is zero for all \( i \geq 0 \). In fact, this map is trivially zero since the inclusion \( \overline{\mathfrak{m}}^{2j-2} \subseteq \mathfrak{m}^l \) (given by the inequality \( l \leq 2j - 2 \)) implies \( \overline{\mathfrak{m}}^{2j-2} = 0 \). Hence \( \rho_l \) is Golod by 2.2. \( \square \)
Lemma 2.4. Let \((R, m, k)\) be a local ring. If an integer \(t\) satisfies \(2 \leq t \leq v(R)\), then \(R/m^{t-1}\) is inert by \(\rho_t\).

Proof. We may assume that \(R\) is complete. Let \(R = Q/I\) be a minimal Cohen presentation, with \((Q, n, k)\) a regular local ring. Since \(t \leq v(R)\), we have \(I \subseteq n^t\). We can make thus the identification \(R/m^j = Q/n^j\) for all \(j \leq t\). Let \(\pi : Q \to R\) and \(\alpha_t : Q \to Q/n^t\) denote the canonical projections. Since \(\alpha_t = \rho_t \circ \pi\), 1.7.1 shows that it suffices to prove that \(R/m^{t-1}\) is inert by \(\alpha_t\). This can be seen by applying 1.7.2 with \(J = n^t, S = Q/n^t, \pi = n/n^t\) and \(M = S/\pi^{t-1} = Q/n^{t-1}\). \(\square\)

Theorem 2.5. Let \((R, m, k)\) be a local ring and let \(t\) be an integer satisfying \(2 \leq t \leq v(R)\). The following are equivalent:

1. \(m^t\) is a small submodule of \(m^{t-1}\);
2. \(\rho_t\) is small;
3. \(\rho_j\) is small for all \(j \geq t\);
4. \(\rho_t\) is Golod;
5. \(\rho_j\) is Golod for all \(j\) such that \(t \leq j \leq 2t - 2\);
6. \(R\) is \(t\)-homogeneous and the algebra \(\text{Ext}^*_R(k, k)\) is generated by \(\text{Ext}^1_R(k, k)\) and \(\text{Ext}^2_R(k, k)\).

Proof. The homological properties under consideration are invariant under completion. We may assume thus \(R\) is complete. Hence \(R = Q/I\) with \((Q, n, k)\) a regular local ring and \(I \subseteq n^t\), with \(t \geq 2\). In particular, we can make the identification \(R/m^t = Q/n^t\).

(2)\(\Rightarrow\)(3): This follows immediately from the definition of small homomorphism.

(3)\(\Rightarrow\)(4): Since \(R/m^t = Q/n^t\) is Golod (see 1.5), we can apply 1.6(2).

(4)\(\Rightarrow\)(2): See 1.6(1).

(4)\(\Rightarrow\)(1): By assumption \(\rho_t\) is Golod. By Lemma 2.4, \(R/m^{t-1}\) is inert by \(\rho_t\). Hence \(m^t\) is a small submodule of \(m^{t-1}\) by Proposition 2.3.

(1)\(\Rightarrow\)(5): See Proposition 2.3.

(5)\(\Rightarrow\)(4): Clear.

(2)\(\Rightarrow\)(6): Assume \(\rho_t\) is small, hence \(\text{Ext}^*_R(k, k)\) is a surjective homomorphism of graded algebras. In [8, 5.9], Levin shows \(\text{Ext}^{1}_{Q/n^t}(k, k)\) is generated by elements in degree 1 and 2. It follows that \(\text{Ext}^*_R(k, k)\) is also generated in degrees 1 and 2. To see that \(R\) is \(t\)-homogeneous, use Remark 1.1(2).

(6)\(\Rightarrow\)(2): Assume \(R\) is \(t\)-homogeneous and the Yoneda algebra \(\text{Ext}^*_R(k, k)\) is generated by \(\text{Ext}^1_R(k, k)\) and \(\text{Ext}^2_R(k, k)\). To show that \(\text{Ext}^*_R(k, k)\) is surjective, it suffices to show that \(\text{Ext}^1_R(k, k)\) and \(\text{Ext}^2_R(k, k)\) are surjective. Since \(t \geq 2\), we have that \(\text{Ker}(\rho_t) \subseteq m^2\), hence \(\text{Ext}^1_R(k, k)\) is an isomorphism, as discussed in 1.6. The fact that \(\text{Ext}^2_R(k, k)\) is surjective is given by Remark 1.1(2). \(\square\)

We say that \(R\) is a complete intersection if the ideal \(I\) in a minimal Cohen presentation \(\widehat{R} = Q/I\) is generated by a regular sequence. For such rings, the structure of the algebra \(\text{Ext}^*_R(k, k)\) is known, see Sjödin [16, §4]. In particular, it is known that this algebra is generated in degrees 1 and 2.

Corollary 2.6. If \(R\) is a \(t\)-homogeneous complete intersection, then conditions (1)-(5) of the Theorem hold. \(\square\)
Remark 2.7. Connected $k$-algebras satisfying the condition that the Yoneda algebra is generated in degrees 1 and 2 are called $K_2$ algebras by Cassidy and Shelton [4]. Since Koszul algebras are characterized by the fact that their Yoneda algebras are generated in degree 1, the notion of $K_2$ algebra can be thought of as a generalization of the notion of Koszul algebra.

3. Compressed Gorenstein local rings

Compressed Gorenstein local rings have been recently studied by Rossi and Şega [13]; we recall below the definition given there. We consider this large class of rings as a case study for the homological properties of interest.

3.1. Compressed Gorenstein local rings. Let $(R, m, k)$ be a Gorenstein artinian local ring. The embedding dimension of $R$ is the integer $e = \text{rank}_k(m/m^2)$, and the socle degree of $R$ is the integer $s$ such that $m^s \neq 0 = m^{s+1}$. Since $R$ is complete, a minimal Cohen presentation of $R$ is $R = Q/I$ with $(Q, n, k)$ a regular local ring and $I \subseteq n^2$. Set

$$\varepsilon_i = \min \left\{ \left( e - 1 + s - i \right), \left( e - 1 + i \right) \right\} \quad \text{for all } i \text{ with } 0 \leq i \leq s.$$  

According to [13, 4.2], we have

$$\lambda(R) \leq \sum_{i=0}^{e} \varepsilon_i,$$

where $\lambda(R)$ denotes the length of $R$. We say that $R$ is a compressed Gorenstein local ring of socle degree $s$ and embedding dimension $e$ if $R$ has maximal length, that is, equality holds in (3.1.1).

If $R$ as above is compressed, we set

$$t = \left\lceil \frac{s + 1}{2} \right\rceil \quad \text{and} \quad r = s + 1 - t,$$

where $\lceil x \rceil$ denotes the smallest integer not less than a rational number $x$.

As discussed in [13, 4.2], we have $t = v(R)$. Note that if $s$ is even then $s = 2t - 2$ and $r = t - 1$. If $s$ is odd then $s = 2t - 1$ and $r = t$.

Remark 3.2. It is shown in [13, 4.2(c)] that if $R$ is a compressed Gorenstein local ring, then $R^e$ is Gorenstein, and it is thus a compressed Gorenstein $k$-algebra. Note that compressed Gorenstein algebras can be regarded as being generic Gorenstein algebras, see the discussion in [13, 5.5].

Let $R$ be a compressed Gorenstein local ring of socle degree $s$. When $s$ is even, the minimal free resolution of $R^e$ over $Q^e$ is described for example by Iarrobino in [6, 4.7]; in particular, it follows that $I^*$ is generated by homogeneous polynomials of degree $t$. According to Lemma 1.3, it follows that $R$ is $t$-homogeneous.

When $s$ is odd, $I^*$ can be generated in degrees $t$ and $t + 1$; see [3, Proposition 3.2]. It is conjectured in [3, 3.13] that $I^*$ is generated in degree $t$, and thus it is $t$-homogeneous, when $R^e$ is generic in a stronger sense.

For the remainder of the section we use the assumptions and notation below.

3.3. Let $(R, m, k)$ be a compressed Gorenstein local ring of embedding dimension $e$ and socle degree $s$, with $2 \leq s \neq 3$. We consider a minimal Cohen presentation $R = Q/I$ with $(Q, n, k)$ a regular local ring and $I \subseteq n^2$. Since $t = v(R)$ we have
I \subseteq n^t and I \nsubseteq n^{t+1}. Let h \in I \cap n^t \setminus n^{t+1}. Set \( P = Q/(h) \) and p = n/(h). Let 
\[ \kappa : P \rightarrow R \]
denote the canonical projection. The following properties shown in \[13\]
will be useful for our approach:
(1) \( m^{t+1} \) is a small submodule of \( m^r \) ([13, Theorem 3.3]);
(2) \( R/m^r \) is a Golod ring for \( 2 \leq j \leq s \) ([13, Proposition 6.3]);
(3) \( \kappa : P \rightarrow R \) is a Golod homomorphism ([13, Theorem 5.1]).
(4) \( P_R(z) \cdot d_R(z) = P_R^Q(z) \) (see [13, Theorem 5.1]), where
\[ d_R(z) = 1 - z(P_R^Q(z) - 1) + z^{e+1}(1 + z). \]

Remark 3.4. Let \( \eta : Q \rightarrow P \) denote the canonical projection. If \( M \) is an \( R \)-module
with \( m^{t-1}M = 0 \), then 1.7.2 shows that \( M \) is inert by \( \kappa \circ \eta \), since \( I \subseteq n^t \). It follows that
\( M \) is also inert by \( \kappa \), by 1.7.1.

Note that the condition \( m^{t-1}M = 0 \) is satisfied for \( M = R/m^j \) with \( j \leq t-1 \)
and also for \( M = m^r \) with \( j \geq r+1 \) (since \( t-1 + r+1 = s+1 \)), and thus \( M \) is
inert by \( \kappa \) and by \( \kappa \circ \eta \), by the above. The case \( M = m^r \) is treated below.

Lemma 3.5. The \( R \)-module \( m^r \) is inert by \( \kappa \).

Proof. Let \( i \geq 0 \). Consider the commutative diagram:

\[
\begin{array}{ccc}
\text{Tor}^P_1(m^{r+1}, k) & \xrightarrow{\text{Tor}^P_1(\nu_{r+1}, k)} & \text{Tor}^P_r(m^r, k) \\
\downarrow{\beta} & & \downarrow{\gamma} \\
\text{Tor}^R_1(m^{r+1}, k) & \xrightarrow{\beta} & \text{Tor}^R_r(m^r, k) \\
\end{array}
\]

where \( \beta = \text{Tor}^P_1(m^r, k) \) and \( \gamma = \text{Tor}^P_1(m^r/m^{r+1}, k) \). Since \( \text{Tor}^P_1(\nu_{r+1}, k) = 0 \) by
3.3(1), \( \alpha \) is injective. Since \( \kappa \) is Golod, it is in particular small by 1.6(1), and
it follows that \( \gamma \) is injective, since \( m^r/m^{r+1} \) is a direct sum of copies of \( k \). The
commutative square on the right shows that \( \beta \) is injective as well, hence \( m^r \) is inert
by \( \kappa \) by 1.7.

We now prove Theorem 2 in the introduction. We restate it below, with some
more detail in part (1).

Theorem 3.6. Let \( 2 \leq s \neq 3 \) and let \( R \) be a compressed Gorenstein local ring of
socle degree \( s \). Let \( n \geq 1 \). The following hold:
(1) \( m^r \) is a small submodule of \( m^{n-1} \) if and only if \( n > s \) or \( n = r+1 \). Further-
more, if \( n \neq r+1 \) and \( n \leq s \), then \( \text{Tor}_1^R(\nu_n, k) \neq 0 \) for infinitely many
values of \( i \).
(2) \( \rho_n : R \rightarrow R/m^n \) is Golod if and only if \( n \geq r+1 \).

Corollary 3.7. With \( R \) as in the theorem, the following hold:
(1) If \( s \) is even, then \( \Ext^n_R(k, k) \) is generated by \( \Ext^n_1(k, k) \) and \( \Ext^n_2(k, k) \).
(2) If \( s \) is odd and \( R \) is \( t \)-homogeneous, then \( \Ext^n_R(k, k) \) is not generated by
\( \Ext^n_1(k, k) \) and \( \Ext^n_2(k, k) \).

Proof. If \( s \) is even, then \( r = t-1 \) and Theorem 3.6(1) gives that \( m^t \) is a small
submodule of \( m^{r-1} \). If \( s \) is odd, then \( r = t \), and Theorem 3.6(1) gives that
Since we know that \( \text{Tor}_r^R \) is a Golod homomorphism, and furthermore a small homomorphism (see Section 2). Let \( n \geq r + 1 \). The homomorphism \( \rho_n \) factors through \( \rho_{r+1} \), and it is thus small as well. Since \( R/m^j \) is Golod by 3.3(2), it follows by (2) in 1.6 that \( \rho_j \) is Golod. If \( n \leq r \), then we also have \( n \leq t \), since \( r = t \) or \( r = t - 1 \). In view of Theorem 2.5, the fact that \( \text{Tor}_n^R(\nu_n, k) \neq 0 \) in this case implies that \( \rho_n \) is not Golod.

We now prove (1). Let \( j \geq 0 \). We use the notation introduced in 1.8, noting that \( \nu_m = \nu_{j+1} \). We have

\[
\text{Tor}_i^R(\nu_{j+1}, k) = 0 \iff T_{m^i}^R(z) = 0
\]

and \( \text{Tor}_i^R(\nu_{j+1}, k) \neq 0 \) for infinitely many \( i \) if and only if \( T_{m^i}^R(z) \notin \mathbb{Z}[z] \).

The conclusion will be established through a concrete computation of \( T_{m^i}^R(z) \).

Using (1.8.1) we have:

\[
(3.7.1) \quad P_{m^i}^S(z) - a_j P_{m^i+j}^S(z) + z P_{m^i+j+1}^S(z) = (1 + z)T_{m^i}^S(z)
\]

where \( a_j = \text{rank}_k(m^j/m^{j+1}) \) and \( S = R \) or \( S = P \) or \( S = Q \). There are four distinct cases to be considered:

Case 1. Assume \( j = r \). Recall that \( m^{r+1} \) and \( m^r \) and \( k \) are all inert by \( \alpha \), by 3.5 and 3.4. Using the definition of inertness for each of these modules, an application of the formula (3.7.1) for \( j = r \), with \( S = R \) and then with \( S = P \), gives:

\[
T_{m^r}^R(z) = T_{m^r}^P(z) \cdot \frac{P_{k}^R(z)}{P_{k}^P(z)}
\]

Since we know that \( \text{Tor}_i^P(\nu_{r+1}, k) = 0 \), see 3.3(1), we have that \( T_{m^i}^P(z) = 0 \), hence \( T_{m^i}^R(z) = 0 \) and thus \( \text{Tor}_i^R(\nu_{r+1}, k) = 0 \).

Case 2. Assume \( r + 1 \leq j < s \). We know that \( m^{j+1}, m^j \) and \( k \) are all inert by \( \alpha \odot \eta: Q \to R \) by 3.4. Proceeding as above, we obtain:

\[
T_{m^i}^R(z) = T_{m^i}^Q(z) \cdot \frac{P_{k}^R(z)}{P_{k}^Q(z)} = \frac{T_{m^i}^Q(z)}{dR(z)}
\]

where the second equality is obtained using 3.3(4).

In [13, Lemma 4.4] it is proved that the map \( \text{Tor}_i^Q(\nu_{r+1}, k) \) is zero for all \( i \neq e \) and is bijective for \( i = e \). The argument given in the proof there, with a minor adjustment, shows that the following more general statement holds: For any \( j \) with \( r \leq j \leq s \), the map \( \text{Tor}_i^Q(\nu_{j+1}, k) \) is zero for all \( i \neq e \) and is bijective for \( i = e \).

Note that \( \text{Tor}_i^Q(m^j, k) \cong \text{Soc}(m^j) \), the socle of \( m^j \). Since \( R \) is Gorenstein, \( \text{rank}_k \text{Soc}(m^j) = \text{rank}_k \text{Soc}(R) = 1 \). It follows that

\[
T_{m^i}^Q(z) = z^e
\]

and thus \( T_{m^i}^R(z) \) is a quotient of a polynomial of degree \( e \) by a polynomial of degree \( e + 2 \). We conclude that \( T_{m^i}^R(z) \) is not a polynomial and thus \( \text{Tor}_i^R(\nu_{j+1}, k) \neq 0 \) for infinitely many \( i \). (On the other hand, note that \( T_{m^i}^R(z) \) is a multiple of \( z^e \) in \( \mathbb{Z}[[z]] \), and this implies that \( \text{Tor}_i^R(\nu_{j+1}, k) = 0 \) for all \( i < e \).)
Case 3. Assume $j = t - 1$ and $j < r$. Since $r = t - 1$ when $s$ is even, this case can happen only when $s$ is odd. In this case, one has $r = t$, hence $j = r - 1$ as well.

In particular, $j + 1 = r$ and we use the already established fact that $T_{m_i}^R(z) = 0$ in the second line below, in order to replace $P_{m_{j+1}}^R(z)$.

$$z(1 + z)T_{m_{j+1}}^R(z) = zP_{m_{j+1}}^R(z) - a_jzP_k^R(z) + z^2P_{m_{j+1}}^R(z)$$

$$= (P_{R/m_{j+1}}^R(z) - 1) - a_jzP_k^R(z) + z^2(a_{j+1}P_k^R(z) - zP_{m_{j+1}}^R(z))$$

$$= \left(P_{R/m_{j+1}}^Q(z) - (a_jz - a_{j+1}z^2)P_k^Q(z) - z^3P_{m_{j+1}}^Q(z)\right) \cdot \frac{P_k^R(z)}{P_k^Q(z)} - 1$$

For the last equality, we have used the definition of inertness and the fact that the $R$-modules $R/m^3$, $m^{j+2}$ and $k$ are all inert by $\times \circ \eta$; this can be seen using again 3.4, since $j = t - 1$ and $j + 2 = r + 1$. Using 3.3(4), we have

$$T_{m_{j+1}}^R(z) = \frac{P_{R/m_{j+1}}^Q(z) - (a_jz - a_{j+1}z^2)P_k^Q(z) - z^3P_{m_{j+1}}^Q(z) - d_{R}(z)}{z(z + 1)\delta_{R}(z)}$$

Since $d_{R}(z)$ has degree $e + 2$ and $P_{m_{j+1}}^Q(z)$ is a polynomial of degree $e$ (note that $m^{j+2} = m^{j+1} \neq 0$), the outcome of this computation is that $T_{m_{j+1}}^R(z)$ is a quotient of a polynomial of degree $e + 3$ by a polynomial of degree $e + 4$. Again, it is clear that $T_{m_{j+1}}^R(z)$ cannot be a polynomial.

Case 4. Assume $j \leq t - 2$. We have:

$$z(1 + z)T_{m_{j+1}}^R(z) = zP_{m_{j+1}}^R(z) - a_jzP_k^R(z) + z^2P_{m_{j+1}}^R(z)$$

$$= (P_{R/m_{j+1}}^R(z) - 1) - a_jzP_k^R(z) + z\left(P_{R/m_{j+1}}^R(z) - 1\right)$$

$$= \left(P_{R/m_{j+1}}^Q(z) - a_jzP_k^Q(z) + zP_{R/m_{j+1}}^Q(z)\right) \cdot \frac{P_k^R(z)}{P_k^Q(z)} - 1 - z$$

where the third equality is due to the fact that $R/m^3$, $R/m^{j+1}$ and $k$ are all inert by $\times \circ \eta$, in view of 3.4. Using 3.3(4) as above, one sees that $T_{m_{j+1}}^R(z)$ can be written as a quotient of a polynomial of degree $e + 3$ by a polynomial of degree $e + 4$, and thus it is not a polynomial.

Since $\text{Tor}_{s}^R(\nu_j, k) = 0$ is clearly zero when $j > s$, we exhausted all cases for $j$. \hfill \Box

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References


Justin Hoffmeier, Department of Mathematics and Statistics, Northwest Missouri State University, Maryville, MO 64468, U.S.A.
E-mail address: jhoff@nwmissouri.edu

Liana M. Šega, Department of Mathematics and Statistics, University of Missouri, Kansas City, MO 64110, U.S.A.
E-mail address: segal@umkc.edu