CONDITIONS FOR THE YONEDA ALGEBRA OF A LOCAL RING TO BE GENERATED IN LOW DEGREES

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Abstract. The powers $\mathfrak{m}^n$ of the maximal ideal $\mathfrak{m}$ of a local Noetherian ring R are known to satisfy certain homological properties for *large* values of n. For example, the homomorphism $R \to R/\mathfrak{m}^n$ is Golod for $n \gg 0$. We study when such properties hold for *small* values of n, and we make connections with the structure of the Yoneda Ext algebra, and more precisely with the property that the Yoneda algebra of R is generated in degrees 1 and 2. A complete treatment of these properties is pursued in the case of compressed Gorenstein local rings.

Introduction

Let $(R, \mathfrak{m}, k)$ be a *local ring*, that is, a commutative noetherian ring R with unique maximal ideal $\mathfrak{m}$ and $k = R/\mathfrak{m}$. For $n \geq 1$ we let $\nu_n \colon \mathfrak{m}^n \to \mathfrak{m}^{n-1}$ denote the canonical inclusion and for each $i \geq 0$ we consider the induced maps

$$\operatorname{Tor}_i^R(\nu_n, k) \colon \operatorname{Tor}_i^R(\mathfrak{m}^n, k) \to \operatorname{Tor}_i^R(\mathfrak{m}^{n-1}, k)\,.$$

Using the terminology of [2], we say that $\mathfrak{m}^n$ is a *small* submodule of $\mathfrak{m}^{n-1}$ if $\operatorname{Tor}_i^R(\nu_n, k) = 0$ for all $i \geq 0$. This condition implies that the canonical projection $\rho_n \colon R \to R/\mathfrak{m}^n$ is a Golod homomorphism, but the converse may not hold.

Levin [9] showed that $\mathfrak{m}^n$ is a small submodule of $\mathfrak{m}^{n-1}$ for all sufficiently large values of n. On the other hand, the fact that $\mathfrak{m}^n$ is a small submodule of $\mathfrak{m}^{n-1}$ for small values of n is an indicator of strong homological properties. It is known that $\mathfrak{m}^2$ is a small submodule of $\mathfrak{m}$ if and only if the Yoneda algebra $\operatorname{Ext}_R(k,k)$ is generated in degree 1, cf. [12, Corollary 1]. More generally, we show:

Theorem 1. *Let $(R, \mathfrak{m}, k)$ be a local ring. Let $\widehat{R} = Q/I$ be a minimal Cohen presentation of R, with $(Q, \mathfrak{n}, k)$ a regular local ring and $I \subseteq \mathfrak{n}^2$. Let t be an integer such that $I \subseteq \mathfrak{n}^t$. The following statements are then equivalent:*

1. *$\mathfrak{m}^t$ is a small submodule of $\mathfrak{m}^{t-1}$;*
2. *$\rho_t \colon R \to R/\mathfrak{m}^t$ is Golod;*
3. *$\rho_n \colon R \to R/\mathfrak{m}^n$ is Golod for all n such that $t \leq n \leq 2t-2$;*
4. *$I \cap \mathfrak{n}^{t+1} \subseteq \mathfrak{n}I$ and the algebra $\operatorname{Ext}_R^*(k,k)$ is generated by $\operatorname{Ext}_R^1(k,k)$ and $\operatorname{Ext}_R^2(k,k)$.*

If R is artinian, its *socle degree* is the largest integer s with $\mathfrak{m}^s \neq 0$. When R is a compressed Gorenstein local ring (see Section 3 for a definition) of socle degree $s \neq 3$, we determine all values of the integer n for which the homomorphism

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ρ_n is Golod, respectively for which $\mathfrak{m}^n$ is a small submodule of $\mathfrak{m}^{n-1}$, and we use Theorem 1 to establish part (3) below.

Theorem 2. *Let $(R,\mathfrak{m},k)$ be a compressed Gorenstein local ring of socle degree s. Assume $2 \le s \ne 3$ and let t denote the smallest integer such that $2t \ge s+1$. If $n \ge 1$, then the following hold:*

1. *$\mathfrak{m}^n$ is a small submodule of $\mathfrak{m}^{n-1}$ if and only if $n > s$ or $n = s+2-t$.*
2. *$\rho_n\colon R \to R/\mathfrak{m}^n$ is Golod if and only if $n \ge s+2-t$.*
3. *If s is even, then $\operatorname{Ext}_R(k,k)$ is generated by $\operatorname{Ext}^1_R(k,k)$ and $\operatorname{Ext}^2_R(k,k)$.*

The conclusion of (3) does not hold when s is odd, see Corollary 3.7.

Section 1 provides definitions and properties of the homological notions of interest. Theorem 1 is proved in Section 2 and Theorem 2 is proved in Section 3.

1. Preliminaries

Throughout the paper $(R,\mathfrak{m},k)$ denotes a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let M be a finitely generated R-module.

We denote by $\widehat{R}$ the completion of R with respect to $\mathfrak{m}$. A *minimal Cohen presentation* of R is a presentation $\widehat{R} = Q/I$, with Q a regular local ring with maximal ideal $\mathfrak{n}$ and I an ideal with $I \subseteq \mathfrak{n}^2$. We know that such a presentation exists, by the Cohen structure theorem.

We denote by R^{g} the associated graded ring with respect to $\mathfrak{m}$, and by M^{g} the associated graded module with respect to $\mathfrak{m}$. We denote by $(R^{\mathsf{g}})_j$ the j-th graded component of R^{g}. For any $x \in R$ we denote by x^* the image of x in $\mathfrak{m}^j/\mathfrak{m}^{j+1} = (R^{\mathsf{g}})_j$, where j is such that $x \in \mathfrak{m}^j \smallsetminus \mathfrak{m}^{j+1}$. For an ideal J of R, we denote by J^* the homogeneous ideal generated by the elements x^* with $x \in J$.

Remark 1.1. With $\widehat{R} = Q/I$ as above, the following then hold:

1. $I \subseteq \mathfrak{n}^t$ if and only if $\operatorname{rank}_k(\mathfrak{m}^{t-1}/\mathfrak{m}^t) = \binom{e+t-2}{e-1}$, where e denotes the minimal number of generators of $\mathfrak{m}$.
2. Assume $t \ge 2$ and $I \subseteq \mathfrak{n}^t$. Then $I \cap \mathfrak{n}^{t+1} \subseteq \mathfrak{n}I$ if and only if the map
$$\operatorname{Ext}^2_{\rho_t}(k,k)\colon \operatorname{Ext}^2_{R/\mathfrak{m}^t}(k,k) \to \operatorname{Ext}^2_R(k,k)$$
induced by the canonical projection $\rho_t\colon R \to R/\mathfrak{m}^t$ is surjective.

To prove (1), note that $(\widehat{R})^{\mathsf{g}} = Q^{\mathsf{g}}/I^*$ and Q^{g} is isomorphic to a polynomial ring over k in e variables of degree 1. We have that $I \subseteq \mathfrak{n}^t$ if and only if $I^* \subseteq (\mathfrak{n}^{\mathsf{g}})^t$, which is equivalent to $(Q^{\mathsf{g}})_{t-1} = (Q^{\mathsf{g}}/I^*)_{t-1}$ and thus to
$$\operatorname{rank}_k(Q^{\mathsf{g}})_{t-1} = \operatorname{rank}_k(\widehat{R}^{\mathsf{g}})_{t-1}\,.$$
Therefore, (1) follows by noting that $\operatorname{rank}_k(Q^{\mathsf{g}})_j = \binom{e-1+j}{e-1}$ and $\operatorname{rank}_k(\widehat{R}^{\mathsf{g}})_j = \operatorname{rank}_k(\mathfrak{m}^j/\mathfrak{m}^{j+1})$ for each j.

For a proof of (2), see Şega [15, 4.3], noting that the map $\operatorname{Ext}^2_{\rho_t}(k,k)$ is surjective if and only if the induced map
$$\operatorname{Tor}_2^{\rho_t}(k,k)\colon \operatorname{Tor}_2^R(k,k) \to \operatorname{Tor}_2^{R/\mathfrak{m}^t}(k,k)$$
is injective.

Definition. Let $\widehat{R} = Q/I$ be a minimal Cohen presentation of R. Let $t \geq 2$ be an integer. We say that the local ring R is *t-homogeneous* if $I \subseteq \mathfrak{n}^t$ and $I \cap \mathfrak{n}^{t+1} \subseteq \mathfrak{n}I$. Remark 1.1 shows that this definition does not depend on the choice of the minimal Cohen presentation.

We set

$$v(R) = \sup\{t \geq 0 \mid I \subseteq \mathfrak{n}^t\}\,.$$

Note that, if R is t-homogeneous and $I \neq 0$, then $t = v(R)$.

Remark 1.2. The terminology of *two-homogeneous algebra* was previously used by Löfwall [11], with a different meaning, in the context of augmented graded algebras.

Lemma 1.3. *Let $\widehat{R} = Q/I$ be a minimal Cohen presentation of R. If the ideal I^* of the polynomial ring Q^{g} is generated by homogeneous polynomials of degree t, then the ring R is t-homogeneous.*

Proof. Assume I^* is generated by homogeneous polynomials of degree t. In particular, it follows that $I \subseteq \mathfrak{n}^t$. To prove $I \cap \mathfrak{n}^{t+1} \subseteq \mathfrak{n}I$, we will show $I \cap \mathfrak{n}^{t+1} \subseteq \mathfrak{n}I + \mathfrak{n}^j$ for all $j \gg 0$. The Krull intersection theorem then gives the conclusion.

Let $x \in I \cap \mathfrak{n}^{t+1}$ and let $a \geq 1$ be the smallest integer such that $x \notin \mathfrak{n}^{t+1+a}$. Then x^* is an element of degree $t + a$ of I^*. Since I^* is generated by homogeneous elements of degree t, we can write

$$x^* = \sum y_i^* z_i^*$$

with $y_i^* \in (Q^{\mathrm{g}})_a$ and $z_i^* \in I^* \cap (Q^{\mathrm{g}})_t$ for each i, where $y_i \in \mathfrak{n}^a$ and $z_i \in I \cap \mathfrak{n}^t$. Set

$$x_1 = x - \sum y_i z_i$$

and note that $x_1 \in I \cap \mathfrak{n}^{t+a+1}$ and $x - x_1 \in \mathfrak{n}I$. In particular $x \in \mathfrak{n}I + \mathfrak{n}^{t+a+1}$. Applying the argument above to x_1, we obtain an element a_1 such that $a_1 > a$, and an element x_2 such that $x_2 \in I \cap \mathfrak{n}^{t+a_1+1}$ and $x_1 - x_2 \in \mathfrak{n}I$. In particular x_1, and thus x, are elements of $\mathfrak{n}I + \mathfrak{n}^{t+a_1+1}$. An inductive argument produces a sequence of integers $1 < a_1 < a_2 < \dots$ such that $x \in \mathfrak{n}I + \mathfrak{n}^{t+a_i+1}$ for all i, and gives the desired conclusion. ☐

Remark 1.4. The converse of the lemma does not hold. This can be seen by considering the 2-homogeneous local ring $R = k[[x,y]]/(x^2 + y^3, xy)$, for which $R^{\mathrm{g}} = k[x,y]/(x^2, y^4, xy)$.

We now proceed to provide definitions for the homological notions of interest, and recall some of their properties.

The *Poincaré series* $\mathrm{P}^R_M(z)$ of M is the formal power series

$$\mathrm{P}^R_M(z) = \sum_{i \geq 0} \mathrm{rank}_k(\mathrm{Tor}^R_i(M,k))z^i.$$

1.5. *Golod rings, modules, and homomorphisms.* Let $(S, \mathfrak{s}, k)$ be a local ring and $\varphi\colon R \to S$ be a surjective homomorphism of local rings. Following Levin [10], we say that an S-module M is *φ-Golod* if the following equality is satisfied:

$$\mathrm{P}^S_M(z) = \frac{\mathrm{P}^R_M(z)}{(1 - z(\mathrm{P}^R_S(z) - 1))}.$$

We say that φ is a *Golod homomorphism* if k is a φ-Golod module.

The ring R is said to be a *Golod ring* if the canonical projection $Q \to \widehat{R}$ is a Golod homomorphism, where $\widehat{R} = Q/I$ is a minimal Cohen presentation. This definition is independent of the choice of representation by [1, Lemma 4.1.3]. Note that this definition is equivalent to the definition given by Avramov in terms of Koszul homology in [1, §5]. A classical example of a Golod ring is $Q/\mathfrak{n}^j$ for any $j \geq 2$, where $(Q, \mathfrak{n}, k)$ is a regular local ring, as first observed by Golod [5] .

1.6. *Small homomorphisms.* Let $\varphi : R \to S$ be a surjective ring homomorphism as above and consider the induced maps

$$\mathrm{Ext}^i_\varphi(k,k) : \mathrm{Ext}^i_S(k,k) \to \mathrm{Ext}^i_R(k,k)$$
$$\mathrm{Tor}^\varphi_i(k,k) : \mathrm{Tor}^R_i(k,k) \to \mathrm{Tor}^S_i(k,k).$$

We say that φ is *small* if $\mathrm{Ext}^*_\varphi(k,k)$ is surjective or, equivalently, if $\mathrm{Tor}^\varphi_*(k,k)$ is injective. Note that $\mathrm{Tor}^\varphi_1(k,k)$ can be identified with the canonical map

$$\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{s}/\mathfrak{s}^2\,.$$

induced by φ. Thus $\mathrm{Tor}^\varphi_1(k,k)$ is an isomorphism if and only if $\mathrm{Ker}(\varphi) \subseteq \mathfrak{m}^2$.

For convenience we collect below a few known facts about small homomorphisms:

(1) If φ is Golod then φ is small, see Avramov [2, 3.5].
(2) If S is a Golod ring and φ is small, then φ is Golod, see Şega [14, 6.7].

1.7. *Inert modules.* Let $\varkappa \colon P \to R$ be a surjective homomorphism of local rings. Following Lescot [7], we say that an R-module M is *inert by* $\varkappa$ if the following equality holds:

$$\mathrm{P}^R_k(z)\,\mathrm{P}^P_M(z) = \mathrm{P}^P_k(z)\,\mathrm{P}^R_M(z).$$

If $\varkappa$ is a Golod homomorphism the following are equivalent:

(1) M is $\varkappa$-Golod;
(2) $\mathrm{Tor}^\varkappa_i(M,k)$ is injective for all i;
(3) M is inert by $\varkappa$.

The equivalence of (1) and (3) is a direct consequence of the definitions and the equivalence of (1) with (2) is given by Levin [10, 1.1].

1.7.1. Consider a sequence of surjective homomorphisms of local rings

$$R \xrightarrow{\alpha} S \xrightarrow{\beta} T\,.$$

Lescot [7, Theorem 3.6] shows that a T-module M is inert by $\beta \circ \alpha$ if an only if M is inert by β and M is inert by α, when considered as an S-module.

1.7.2. Let Q be a regular local ring with maximal ideal $\mathfrak{n}$ and an ideal J such that $J \subseteq \mathfrak{n}^t$ with $t \geq 2$. Set $S = Q/J$ and $\overline{\mathfrak{n}} = \mathfrak{n}/J$. If a finitely generated S-module N is annihilated by $\overline{\mathfrak{n}}^{t-1}$ then N is inert by the natural projection $\varkappa : Q \to S$, see [7, 3.7 and 3.11]

We end this section with introducing some more notation.

1.8. If $(R, \mathfrak{m}, k)$ is a local ring and M is a finite R-module, we let $\nu_M \colon \mathfrak{m}M \to M$ denote the canonical inclusion and consider the induced maps

$$\mathrm{Tor}^R_i(\nu_M, k)\colon\ \mathrm{Tor}^R_i(\mathfrak{m}M, k) \to \mathrm{Tor}^R_i(M,k)\,.$$

They fit into the long exact sequence

$$\cdots \to \mathrm{Tor}^R_i(\mathfrak{m}M,k) \xrightarrow{\mathrm{Tor}^R_i(\nu_M,k)} \mathrm{Tor}^R_i(M,k) \to \mathrm{Tor}^R_i(M/\mathfrak{m}M,k) \to \cdots\,.$$

If Λ is a graded vector space over k, we set $H_\Lambda(z)=\sum_{i\geq 0}\operatorname{rank}_k(\Lambda_i)z^i$; this formal power series is called the *Hilbert series* of Λ. Since rank is additive on exact sequences, a rank count in the exact sequence above gives

$$\mathrm{P}^R_M(z)-a\,\mathrm{P}^R_k(z)+z\,\mathrm{P}^R_{\mathfrak{m}M}(z)=(1+z)H_{\operatorname{Im}(\operatorname{Tor}^R_*(\nu_M,k))}(z) \tag{1.8.1}$$

where $a=\operatorname{rank}_k(M/\mathfrak{m}M)$. We set

$$T^R_M(z)=H_{\operatorname{Im}(\operatorname{Tor}^R_*(\nu_M,k))}(z)\,.$$

2. Homological properties of powers of the maximal ideal

The purpose of this section is to prove Theorem 1 in the introduction, restated as Theorem 2.5 below.

For each integer j let

$$\rho_j\colon R\to R/\mathfrak{m}^j \qquad\text{and}\qquad \nu_j\colon \mathfrak{m}^j\to\mathfrak{m}^{j-1}$$

denote the canonical projection, respectively the inclusion, and consider the induced maps

$$\begin{aligned}\operatorname{Tor}^{\rho_j}_i(R/\mathfrak{m}^{j-1},k)\colon\ \operatorname{Tor}^R_i(R/\mathfrak{m}^{j-1},k)&\to\operatorname{Tor}^{R/\mathfrak{m}^j}_i(R/\mathfrak{m}^{j-1},k)\\ \operatorname{Tor}^R_i(\nu_j,k)\colon\ \operatorname{Tor}^R_i(\mathfrak{m}^j,k)&\to\operatorname{Tor}^R_i(\mathfrak{m}^{j-1},k).\end{aligned}$$

Using the terminolgy in the appendix of [2], we say that $\mathfrak{m}^j$ is a *small submodule* of $\mathfrak{m}^{j-1}$ if $\operatorname{Tor}^R_i(\nu_j,k)=0$ for all $i\geq 0$.

Remark 2.1. For $i,j\geq 0$ let

$$\eta^j_i\colon\ \operatorname{Tor}^R_i(R/\mathfrak{m}^j,k)\to\operatorname{Tor}^R_i(R/\mathfrak{m}^{j-1},k)$$

denote the map induced by the canonical projection $R/\mathfrak{m}^j\to R/\mathfrak{m}^{j-1}$.

Note that $\operatorname{Tor}^R_i(\nu_j,k)=0$ if and only if $\eta^j_{i+1}=0$. Indeed, this is a standard argument, using the canonical isomorphisms $\operatorname{Tor}^R_{i+1}(R/\mathfrak{m}^n,k)\cong\operatorname{Tor}^R_i(\mathfrak{m}^n,k)$ which arise as connecting homomorphisms in the long exact sequence associated to the exact sequence

$$0\to\mathfrak{m}^n\to R\to R/\mathfrak{m}^n\to 0\,,$$

with $n=j$ and $n=j-1$.

2.2. We state here a needed result of Rossi and Şega [13, Lemma 1.2]:

Let $\varkappa\colon(P,\mathfrak{p},k)\to(R,\mathfrak{m},k)$ be a surjective homomorphism of local rings. Assume there exists an integer a such that

(1) The map $\operatorname{Tor}^P_i(R,k)\to\operatorname{Tor}^P_i(R/\mathfrak{m}^a,k)$ induced by the natural projection $R\to R/\mathfrak{m}^a$ is zero for all $i>0$.
(2) The map $\operatorname{Tor}^P_i(\mathfrak{m}^{2a},k)\to\operatorname{Tor}^P_i(\mathfrak{m}^a,k)$ induced by the inclusion $\mathfrak{m}^{2a}\hookrightarrow\mathfrak{m}^a$ is zero for all $i\geq 0$.

Then $\varkappa$ is a Golod homomorphism.

Proposition 2.3. *Let $(R,\mathfrak{m},k)$ be a local ring and let $j\geq 2$ be an integer. The following are equivalent:*

(1) *$\mathfrak{m}^j$ is a small submodule of $\mathfrak{m}^{j-1}$;*
(2) *$\operatorname{Tor}^{\rho_j}_i(R/\mathfrak{m}^{j-1},k)$ is injective for all $i\geq 0$;*
(3) *ρ_j is Golod and $R/\mathfrak{m}^{j-1}$ is inert by ρ_j.*

If these conditions hold, then ρ_l is Golod for all integers l with $j\leq l\leq 2j-2$.

Proof. (1)⇒(2): Let $i \geq 0$. Set $\overline{\mathfrak{m}}^{j-1} = \mathfrak{m}^{j-1}/\mathfrak{m}^j$. Consider long exact sequences associated to the exact sequence

$$0 \to \overline{\mathfrak{m}}^{j-1} \to R/\mathfrak{m}^j \to R/\mathfrak{m}^{j-1} \to 0$$

and create the following commutative diagram with exact columns.

$$\begin{array}{ccc}
\operatorname{Tor}^R_{i+1}(R/\mathfrak{m}^j, k) & \longrightarrow & \operatorname{Tor}^{R/\mathfrak{m}^j}_{i+1}(R/\mathfrak{m}^j, k) = 0 \\
\Big\downarrow \eta^j_{i+1} & & \Big\downarrow \\
\operatorname{Tor}^R_{i+1}(R/\mathfrak{m}^{j-1}, k) & \xrightarrow{\operatorname{Tor}^{\rho_j}_{i+1}(R/\mathfrak{m}^{j-1},k)} & \operatorname{Tor}^{R/\mathfrak{m}^j}_{i+1}(R/\mathfrak{m}^{j-1}, k) \\
\Big\downarrow \Delta_i & & \Big\downarrow \\
\operatorname{Tor}^R_i(\overline{\mathfrak{m}}^{j-1}, k) & \xrightarrow{\operatorname{Tor}^{\rho_j}_i(\overline{\mathfrak{m}}^{j-1},k)} & \operatorname{Tor}^{R/\mathfrak{m}^j}_i(\overline{\mathfrak{m}}^{j-1}, k)
\end{array}$$

By Remark 2.1, the hypothesis that $\operatorname{Tor}^R_i(\nu_j, k) = 0$ implies that $\eta^j_{i+1} = 0$, and thus the connecting homomorphism Δ_i is injective.

Levin's proof of [9, 3.15] shows $\operatorname{Tor}^R_i(\nu_j, k) = 0$ for all i implies ρ_j is Golod. (This also follows from the last part of the proof.) In particular, the map ρ_j is small by 1.6(1). Since $\overline{\mathfrak{m}}^{j-1}$ is a direct sum of copies of k, it follows that $\operatorname{Tor}^{\rho_j}_i(\overline{\mathfrak{m}}^{j-1}, k)$ is injective.

The bottom commutative square yields that $\operatorname{Tor}^{\rho_j}_{i+1}(R/\mathfrak{m}^{j-1}, k)$ is injective.

(2)⇒(1): Assuming that $\operatorname{Tor}^{\rho_j}_{i+1}(R/\mathfrak{m}^{j-1}, k)$ is injective, the top square in the commutative diagram above gives that $\eta^j_{i+1} = 0$, and thus $\operatorname{Tor}^R_i(\nu_j, k) = 0$ by Remark 2.1.

(1)⇒(3): As mentioned above, $\operatorname{Tor}^R_i(\nu_j, k) = 0$ for all i implies that ρ_j is Golod. Since we already proved (1)⇒(2), we know that $\operatorname{Tor}^{\rho_j}_i(R/\mathfrak{m}^{j-1}, k)$ is injective for all i. By 1.7, $R/\mathfrak{m}^{j-1}$ is then inert by ρ_j.

(3)⇒(2): see 1.7.

Fix now l such that $j \leq l \leq 2j-2$. We prove the last assertion of the proposition by applying 2.2, with $\varkappa = \rho_l$ and $a = j-1$. Set $\overline{R} = R/\mathfrak{m}^l$ and $\overline{\mathfrak{m}} = \mathfrak{m}/\mathfrak{m}^l$. Let

$$\overline{\rho}_{j-1} \colon \overline{R} \to \overline{R}/\overline{\mathfrak{m}}^{j-1}$$

denote the canonical projection. To satisfy the first hypothesis of 2.2, we will show that the induced map

$$\operatorname{Tor}^R_i(\overline{\rho}_{j-1}, k) : \operatorname{Tor}^R_i(\overline{R}, k) \to \operatorname{Tor}^R_i(\overline{R}/\overline{\mathfrak{m}}^{j-1}, k)$$

is zero for all $i > 0$. Since $l \geq j$, we have $\overline{R}/\overline{\mathfrak{m}}^{j-1} = R/\mathfrak{m}^{j-1}$ and $\operatorname{Tor}^R_i(\overline{\rho}_{j-1}, k)$ factors through

$$\eta^j_i \colon \operatorname{Tor}^R_i(R/\mathfrak{m}^j, k) \to \operatorname{Tor}^R_i(R/\mathfrak{m}^{j-1}, k)\,.$$

Since $\operatorname{Tor}^R_i(\nu_j, k) = 0$ for all $i \geq 0$ by assumption, we have that $\eta^j_i = 0$ for all $i > 0$ by Remark 2.1. Hence $\operatorname{Tor}^R_i(\overline{\rho}_{j-1}, k) = 0$ for all $i > 0$.

To satisfy the second hypothesis of 2.2, we need to show that the induced map

$$\operatorname{Tor}^R_i(\overline{\mathfrak{m}}^{2(j-1)}, k) \to \operatorname{Tor}^R_i(\overline{\mathfrak{m}}^{j-1}, k)$$

induced by the inclusion $\overline{\mathfrak{m}}^{2(j-1)} \hookrightarrow \overline{\mathfrak{m}}^{j-1}$ is zero for all $i \geq 0$. In fact, this map is trivially zero since the inclusion $\mathfrak{m}^{2j-2} \subseteq \mathfrak{m}^l$ (given by the inequality $l \leq 2j-2$) implies $\overline{\mathfrak{m}}^{2j-2} = 0$. Hence ρ_l is Golod by 2.2. □

Lemma 2.4. *Let $(R, \mathfrak{m}, k)$ be a local ring. If an integer t satisfies $2 \le t \le v(R)$, then $R/\mathfrak{m}^{t-1}$ is inert by ρ_t.*

Proof. We may assume that R is complete. Let $R = Q/I$ be a minimal Cohen presentation, with $(Q, \mathfrak{n}, k)$ a regular local ring. Since $t \le v(R)$, we have $I \subseteq \mathfrak{n}^t$. We can make thus the identification $R/\mathfrak{m}^j = Q/\mathfrak{n}^j$ for all $j \le t$. Let $\varkappa : Q \to R$ and $\alpha_t : Q \to Q/\mathfrak{n}^t$ denote the canonical projections. Since $\alpha_t = \rho_t \circ \varkappa$, 1.7.1 shows that it suffices to prove that $R/\mathfrak{m}^{t-1}$ is inert by α_t. This can be seen by applying 1.7.2 with $J = \mathfrak{n}^t$, $S = Q/\mathfrak{n}^t$, $\overline{\mathfrak{n}} = \mathfrak{n}/\mathfrak{n}^t$ and $M = S/\overline{\mathfrak{n}}^{t-1} = Q/\mathfrak{n}^{t-1}$. □

Theorem 2.5. *Let $(R, \mathfrak{m}, k)$ be a local ring and let t be an integer satisfying $2 \le t \le v(R)$. The following are equivalent:*

1. *$\mathfrak{m}^t$ is a small submodule of $\mathfrak{m}^{t-1}$;*
2. *ρ_t is small;*
3. *ρ_j is small for all $j \ge t$;*
4. *ρ_t is Golod;*
5. *ρ_j is Golod for all j such that $t \le j \le 2t-2$;*
6. *R is t-homogeneous and the algebra $\mathrm{Ext}^*_R(k,k)$ is generated by $\mathrm{Ext}^1_R(k,k)$ and $\mathrm{Ext}^2_R(k,k)$.*

Proof. The homological properties under consideration are invariant under completion. We may assume thus R is complete. Hence $R = Q/I$ with $(Q, \mathfrak{n}, k)$ a regular local ring and $I \subseteq \mathfrak{n}^t$, with $t \ge 2$. In particular, we can make the identification $R/\mathfrak{m}^t = Q/\mathfrak{n}^t$.

(2)⇒(3): This follows immediately from the definition of small homomorphim.

(3)⇒(2): Clear.

(3)⇒(4): Since $R/\mathfrak{m}^t = Q/\mathfrak{n}^t$ is Golod (see 1.5), we can apply 1.6(2).

(4)⇒(2): See 1.6(1).

(4)⇒(1): By assumption ρ_t is Golod. By Lemma 2.4, $R/\mathfrak{m}^{t-1}$ is inert by ρ_t. Hence $\mathfrak{m}^t$ is a small submodule of $\mathfrak{m}^{t-1}$ by Proposition 2.3.

(1)⇒(5): See Proposition 2.3.

(5) ⇒(4): Clear.

(2) ⇒(6): Assume ρ_t is small, hence $\mathrm{Ext}_{\rho_t}(k,k)$ is a surjective homomorphism of graded algebras. In [8, 5.9], Levin shows $\mathrm{Ext}_{Q/\mathfrak{n}^t}(k,k)$ is generated by elements in degree 1 and 2. It follows that $\mathrm{Ext}_R(k,k)$ is also generated in degrees 1 and 2. To see that R is t-homogeneous, use Remark 1.1(2).

(6) ⇒(2): Assume R is t-homogeneous and the Yoneda algebra $\mathrm{Ext}_R(k,k)$ is generated by $\mathrm{Ext}^1_R(k,k)$ and $\mathrm{Ext}^2_R(k,k)$. To show that $\mathrm{Ext}_{\rho_t}(k,k)$ is surjective, it suffices to show that $\mathrm{Ext}^1_{\rho_t}(k,k)$ and $\mathrm{Ext}^2_{\rho_t}(k,k)$ are surjective. Since $t \ge 2$, we have that $\mathrm{Ker}(\rho_t) \subseteq \mathfrak{m}^2$, hence $\mathrm{Ext}^1_{\rho_t}(k,k)$ is an isomorphism, as discussed in 1.6. The fact that $\mathrm{Ext}^2_{\rho_t}(k,k)$ is surjective is given by Remark 1.1(2). □

We say that R is a complete intersection if the ideal I in a minimal Cohen presentation $\widehat{R} = Q/I$ is generated by a regular sequence. For such rings, the structure of the algebra $\mathrm{Ext}_R(k,k)$ is known, see Sjödin [16, §4]. In particular, it is known that this algebra is generated in degrees 1 and 2.

Corollary 2.6. *If R is a t-homogeneous complete intersection, then conditions (1)-(5) of the Theorem hold.* □

Remark 2.7. Connected k-algebras satisfying the condition that the Yoneda algebra is generated in degrees 1 and 2 are called $\mathcal{K}_2$ algebras by Cassidy and Shelton [4]. Since Koszul algebras are characterized by the fact that their Yoneda algebras are generated in degree 1, the notion of $\mathcal{K}_2$ algebra can be thought of as a generalization of the notion of Koszul algebra.

3. Compressed Gorenstein local rings

Compressed Gorenstein local rings have been recently studied by Rossi and Şega [13]; we recall below the definition given there. We consider this large class of rings as a case study for the homological properties of interest.

3.1. *Compressed Gorenstein local rings.* Let $(R,\mathfrak{m},k)$ be a Gorenstein artinian local ring. The embedding dimension of R is the integer $e=\operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$, and the socle degree of R is the integer s such that $\mathfrak{m}^s\neq 0=\mathfrak{m}^{s+1}$. Since R is complete, a minimal Cohen presentation of R is $R=Q/I$ with $(Q,\mathfrak{n},k)$ a regular local ring and $I\subseteq\mathfrak{n}^2$. Set

$$\varepsilon_i=\min\left\{\binom{e-1+s-i}{e-1},\binom{e-1+i}{e-1}\right\}\quad\text{for all } i \text{ with } 0\le i\le s.$$

According to [13, 4.2], we have

$$\lambda(R)\le\sum_{i=0}^{e}\varepsilon_i\,, \tag{3.1.1}$$

where $\lambda(R)$ denotes the *length* of R. We say that R is a *compressed* Gorenstein local ring of socle degree s and embedding dimension e if R has maximal length, that is, equality holds in (3.1.1).

If R as above is compressed, we set

$$t=\left\lceil\frac{s+1}{2}\right\rceil\quad\text{and}\quad r=s+1-t\,, \tag{3.1.2}$$

where $\lceil x\rceil$ denotes the smallest integer not less than a rational number x.

As discussed in [13, 4.2], we have $t=v(R)$. Note that if s is even then $s=2t-2$ and $r=t-1$. If s is odd then $s=2t-1$ and $r=t$.

Remark 3.2. It is shown in [13, 4.2(c)] that if R is a compressed Gorenstein local ring, then $R^{\mathfrak{g}}$ is Gorenstein, and it is thus a compressed Gorenstein k-algebra. Note that compressed Gorenstein algebras can be regarded as being generic Gorenstein algebras, see the discussion in [13, 5.5].

Let R be a compressed Gorenstein local ring of socle degree s. When s is even, the minimal free resolution of $R^{\mathfrak{g}}$ over $Q^{\mathfrak{g}}$ is described for example by Iarrobino in [6, 4.7]; in particular, it follows that I^* is generated by homogeneous polynomials of degree t. According to Lemma 1.3, it follows that R is t-homogeneous.

When s is odd, I^* can be generated in degrees t and $t+1$; see [3, Proposition 3.2]. It is conjectured in [3, 3.13] that I^* is generated in degree t, and thus it is t-homogeneous, when $R^{\mathfrak{g}}$ is generic in a stronger sense.

For the remainder of the section we use the assumptions and notation below.

3.3. Let $(R,\mathfrak{m},k)$ be a compressed Gorenstein local ring of embedding dimension e and socle degree s, with $2\le s\neq 3$. We consider a minimal Cohen presentation $R=Q/I$ with $(Q,\mathfrak{n},k)$ a regular local ring and $I\subseteq\mathfrak{n}^2$. Since $t=v(R)$ we have

$I \subseteq \mathfrak{n}^t$ and $I \not\subseteq \mathfrak{n}^{t+1}$. Let $h \in I \cap \mathfrak{n}^t \setminus \mathfrak{n}^{t+1}$. Set $P = Q/(h)$ and $\mathfrak{p} = \mathfrak{n}/(h)$. Let $\varkappa : P \to R$ denote the canonical projection. The following properties shown in [13] will be useful for our approach:

(1) $\mathfrak{m}^{r+1}$ is a small submodule of $\mathfrak{m}^r$ ([13, Theorem 3.3]);
(2) $R/\mathfrak{m}^j$ is a Golod ring for $2 \le j \le s$ ([13, Proposition 6.3]);
(3) $\varkappa : P \to R$ is a Golod homomorphism ([13, Theorem 5.1]).
(4) $\mathrm{P}^R_k(z) \cdot d_R(z) = \mathrm{P}^Q_k(z)$ (see [13, Theorem 5.1]), where

$$d_R(z) = 1 - z(\mathrm{P}^Q_R(z) - 1) + z^{e+1}(1+z).$$

Note that $d_R(z)$ is polynomial of degree $e+2$, since $\mathrm{P}^Q_R(z)$ is a polynomial of degree e.

Remark 3.4. Let $\eta\colon Q \to P$ denote the canonical projection. If M is an R-module with $\mathfrak{m}^{t-1}M = 0$, then 1.7.2 shows that M is inert by $\varkappa \circ \eta$, since $I \subseteq \mathfrak{n}^t$. It follows that M is also inert by $\varkappa$, by 1.7.1.

Note that the condition $\mathfrak{m}^{t-1}M = 0$ is satisfied for $M = R/\mathfrak{m}^j$ with $j \le t-1$ and also for $M = \mathfrak{m}^j$ with $j \ge r+1$ (since $t-1+r+1 = s+1$), and thus M is inert by $\varkappa$ and by $\varkappa \circ \eta$, by the above. The case $M = \mathfrak{m}^r$ is treated below.

Lemma 3.5. *The R-module $\mathfrak{m}^r$ is inert by $\varkappa$.*

Proof. Let $i \ge 0$. Consider the commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Tor}^P_i(\mathfrak{m}^{r+1}, k) & \xrightarrow{\mathrm{Tor}^P_i(\nu_{r+1},k)} & \mathrm{Tor}^P_i(\mathfrak{m}^r, k) & \xrightarrow{\alpha} & \mathrm{Tor}^P_i(\mathfrak{m}^r/\mathfrak{m}^{r+1}, k) \\
\downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\
\mathrm{Tor}^R_i(\mathfrak{m}^{r+1}, k) & \longrightarrow & \mathrm{Tor}^R_i(\mathfrak{m}^r, k) & \longrightarrow & \mathrm{Tor}^R_i(\mathfrak{m}^r/\mathfrak{m}^{r+1}, k)
\end{array}$$

where $\beta = \mathrm{Tor}^{\varkappa}_i(\mathfrak{m}^r, k)$ and $\gamma = \mathrm{Tor}^{\varkappa}_i(\mathfrak{m}^r/\mathfrak{m}^{r+1}, k)$. Since $\mathrm{Tor}^P_i(\nu_{r+1}, k) = 0$ by 3.3(1), α is injective. Since $\varkappa$ is Golod, it is in particular small by 1.6(1), and it follows that γ is injective, since $\mathfrak{m}^r/\mathfrak{m}^{r+1}$ is a direct sum of copies of k. The commutative square on the right shows that β is injective as well, hence $\mathfrak{m}^r$ is inert by $\varkappa$ by 1.7. □

We now prove Theorem 2 in the introduction. We restate it below, with some more detail in part (1).

Theorem 3.6. *Let $2 \le s \ne 3$ and let R be a compressed Gorenstein local ring of socle degree s. Let $n \ge 1$. The following hold:*

(1) *$\mathfrak{m}^n$ is a small submodule of $\mathfrak{m}^{n-1}$ if and only if $n > s$ or $n = r+1$. Furthermore, if $n \ne r+1$ and $n \le s$, then $\mathrm{Tor}^R_i(\nu_n, k) \ne 0$ for infinitely many values of i.*
(2) *$\rho_n\colon R \to R/\mathfrak{m}^n$ is Golod if and only if $n \ge r+1$.*

Corollary 3.7. *With R as in the theorem, the following hold:*

(1) *If s is even, then $\mathrm{Ext}_R(k,k)$ is generated by $\mathrm{Ext}^1_R(k,k)$ and $\mathrm{Ext}^2_R(k,k)$.*
(2) *If s is odd and R is t-homogeneous, then $\mathrm{Ext}_R(k,k)$ is not generated by $\mathrm{Ext}^1_R(k,k)$ and $\mathrm{Ext}^2_R(k,k)$.*

Proof. If s is even, then $r = t-1$ and Theorem 3.6(1) gives that $\mathfrak{m}^t$ is a small submodule of $\mathfrak{m}^{t-1}$. If s is odd, then $r = t$, and Theorem 3.6(1) gives that

$\operatorname{Tor}_i^R(\nu_t, k) \neq 0$ for infinitely many values of i, hence $\mathfrak{m}^t$ is not a small submodule of $\mathfrak{m}^{t-1}$. Both conclusions follow then from Theorem 2.5. □

Proof of Theorem 3.6. Assuming that (1) is proved, we prove (2) as follows.

Since $\mathfrak{m}^{r+1}$ is a small submodule of $\mathfrak{m}^r$, we know that that ρ_{r+1} is a Golod homomorphism, and furthermore a small homomorphism (see Section 2). Let $n \geq r+1$. The homomorphism ρ_n factors thorugh ρ_{r+1}, and it is thus small as well. Since $R/\mathfrak{m}^j$ is Golod by 3.3(2), it follows by (2) in 1.6 that ρ_j is Golod. If $n \leq r$, then we also have $n \leq t$, since $r = t$ or $r = t-1$. In view of Theorem 2.5, the fact that $\operatorname{Tor}_*^R(\nu_n, k) \neq 0$ in this case implies that ρ_n is not Golod.

We now prove (1). Let $j \geq 0$. We use the notation introduced in 1.8, noting that $\nu_{\mathfrak{m}^j} = \nu_{j+1}$. We have

$$\operatorname{Tor}_*^R(\nu_{j+1}, k) = 0 \iff T^R_{\mathfrak{m}^j}(z) = 0$$

and $\operatorname{Tor}_i^R(\nu_{j+1}, k) \neq 0$ for infinitely many i if and only if $T^R_{\mathfrak{m}^j}(z) \notin \mathbb{Z}[z]$.

The conclusion will be established through a concrete computation of $T^R_{\mathfrak{m}^j}(z)$.

Using (1.8.1) we have:

$$\mathrm{P}^S_{\mathfrak{m}^j}(z) - a_j\,\mathrm{P}^S_k(z) + z\,\mathrm{P}^S_{\mathfrak{m}^{j+1}}(z) = (1+z)T^S_{\mathfrak{m}^j}(z) \tag{3.7.1}$$

where $a_j = \operatorname{rank}_k(\mathfrak{m}^j/\mathfrak{m}^{j+1})$ and $S = R$ or $S = P$ or $S = Q$. There are four distinct cases to be considered:

Case 1. Assume $j = r$. Recall that $\mathfrak{m}^{r+1}$ and $\mathfrak{m}^r$ and k are all inert by $\varkappa$, by 3.5 and 3.4. Using the definition of inertness for each of these modules, an application of the formula (3.7.1) for $j = r$, with $S = R$ and then with $S = P$, gives:

$$T^R_{\mathfrak{m}^r}(z) = T^P_{\mathfrak{m}^r}(z) \cdot \frac{\mathrm{P}^R_k(z)}{\mathrm{P}^P_k(z)}\,.$$

Since we know that $\operatorname{Tor}_*^P(\nu_{r+1}, k) = 0$, see 3.3(1), we have that $T^P_{\mathfrak{m}^r}(z) = 0$, hence $T^R_{\mathfrak{m}^r}(z) = 0$ and thus $\operatorname{Tor}_*^R(\nu_{r+1}, k) = 0$.

Case 2. Assume $r+1 \leq j < s$. We know that $\mathfrak{m}^{j+1}$, $\mathfrak{m}^j$ and k are all inert by $\varkappa \circ \eta \colon Q \to R$ by 3.4. Proceeding as above, we obtain:

$$T^R_{\mathfrak{m}^j}(z) = T^Q_{\mathfrak{m}^j}(z) \cdot \frac{\mathrm{P}^R_k(z)}{\mathrm{P}^Q_k(z)} = \frac{T^Q_{\mathfrak{m}^j}(z)}{d_R(z)}$$

where the second equality is obtained using 3.3(4).

In [13, Lemma 4.4] it is proved that the map $\operatorname{Tor}_i^Q(\nu_{r+1}, k)$ is zero for all $i \neq e$ and is bijective for $i = e$. The argument given in the proof there, with a minor adjustment, shows that the following more general statement holds: For any j with $r \leq j \leq s$, the map $\operatorname{Tor}_i^Q(\nu_{j+1}, k)$ is zero for all $i \neq e$ and is bijective for $i = e$.

Note that $\operatorname{Tor}_e^Q(\mathfrak{m}^j, k) \cong \operatorname{Soc}(\mathfrak{m}^j)$, the socle of $\mathfrak{m}^j$. Since R is Gorenstein, $\operatorname{rank}_k \operatorname{Soc}(\mathfrak{m}^j) = \operatorname{rank}_k \operatorname{Soc}(R) = 1$. It follows that

$$T^Q_{\mathfrak{m}^j}(z) = z^e$$

and thus $T^R_{\mathfrak{m}^j}(z)$ is a quotient of a polynomial of degree e by a polynomial of degree $e+2$. We conclude that $T^R_{\mathfrak{m}^j}(z)$ is not a polynomial and thus $\operatorname{Tor}_i^R(\nu_{j+1}, k) \neq 0$ for infinitely many i. (On the other hand, note that $T^R_{\mathfrak{m}^j}(z)$ is a multiple of z^e in $\mathbb{Z}[[z]]$, and this implies that $\operatorname{Tor}_i^R(\nu_{j+1}, k) = 0$ for all $i < e$.)

Case 3. Assume $j = t-1$ and $j < r$. Since $r = t-1$ when s is even, this case can happen only when s is odd. In this case, one has $r = t$, hence $j = r-1$ as well. In particular, $j+1 = r$ and we use the already established fact that $T^R_{\mathfrak{m}^r}(z) = 0$ in the second line below, in order to replace $\mathrm{P}^R_{\mathfrak{m}^{j+1}}(z)$.

$$\begin{aligned} z(1+z)T^R_{\mathfrak{m}^j}(z) &= z\,\mathrm{P}^R_{\mathfrak{m}^j}(z) - a_j z\,\mathrm{P}^R_k(z) + z^2\,\mathrm{P}^R_{\mathfrak{m}^{j+1}}(z) \\ &= (\mathrm{P}^R_{R/\mathfrak{m}^j}(z) - 1) - a_j z\,\mathrm{P}^R_k(z) + z^2\big(a_{j+1}\,\mathrm{P}^R_k(z) - z\,\mathrm{P}^R_{\mathfrak{m}^{j+2}}(z)\big) \\ &= \Big(\mathrm{P}^Q_{R/\mathfrak{m}^j}(z) - (a_j z - a_{j+1}z^2)\,\mathrm{P}^Q_k(z) - z^3\,\mathrm{P}^Q_{\mathfrak{m}^{j+2}}(z)\Big)\cdot\frac{\mathrm{P}^R_k(z)}{\mathrm{P}^Q_k(z)} - 1 \end{aligned}$$

For the last equality, we have used the definition of inertness and the fact that the R-modules $R/\mathfrak{m}^j$, $\mathfrak{m}^{j+2}$ and k are all inert by $\varkappa\circ\eta$; this can be seen using again 3.4, since $j = t-1$ and $j+2 = r+1$. Using 3.3(4), we have

$$T^R_{\mathfrak{m}^j}(z) = \frac{\mathrm{P}^Q_{R/\mathfrak{m}^j}(z) - (a_j z - a_{j+1}z^2)\,\mathrm{P}^Q_k(z) - z^3\,\mathrm{P}^Q_{\mathfrak{m}^{j+2}}(z) - d_R(z)}{z(z+1)d_R(z)}$$

Since $d_R(z)$ has degree $e+2$ and $\mathrm{P}^Q_{\mathfrak{m}^{j+2}}(z)$ is a polynomial of degree e (note that $\mathfrak{m}^{j+2} = \mathfrak{m}^{t+1} \neq 0$), the outcome of this computation is that $T^R_{\mathfrak{m}^j}(z)$ is a quotient of a polynomial of degree $e+3$ by a polynomial of degree $e+4$. Again, it is clear that $T^R_{\mathfrak{m}^j}(z)$ cannot be a polynomial.

Case 4. Assume $j \le t-2$. We have:

$$\begin{aligned} z(1+z)T^R_{\mathfrak{m}^j}(z) &= z\,\mathrm{P}^R_{\mathfrak{m}^j}(z) - a_j z\,\mathrm{P}^R_k(z) + z^2\,\mathrm{P}^R_{\mathfrak{m}^{j+1}}(z) \\ &= (\mathrm{P}^R_{R/\mathfrak{m}^j}(z) - 1) - a_j z\,\mathrm{P}^R_k(z) + z\big(\,\mathrm{P}^R_{R/\mathfrak{m}^{j+1}}(z) - 1\big) \\ &= \Big(\mathrm{P}^Q_{R/\mathfrak{m}^j}(z) - a_j z\,\mathrm{P}^Q_k(z) + z\,\mathrm{P}^Q_{R/\mathfrak{m}^{j+1}}(z)\Big)\cdot\frac{\mathrm{P}^R_k(z)}{\mathrm{P}^Q_k(z)} - 1 - z \end{aligned}$$

where the third equality is due to the fact that $R/\mathfrak{m}^j$, $R/\mathfrak{m}^{j+1}$ and k are all inert by $\varkappa\circ\eta$, in view of 3.4. Using 3.3(4) as above, one sees that $T^R_{\mathfrak{m}^j}(z)$ can be written as a qotient of a polynomial of degree $e+3$ by a polynomial of degree $e+4$, and thus it is not a polynomial.

Since $\mathrm{Tor}^R_*(\nu_j, k) = 0$ is clearly zero when $j > s$, we exhausted all cases for j. □

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