ACYCLIC COMPLEXES OF FINITELY GENERATED FREE MODULES OVER LOCAL RINGS

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Abstract
We consider the question of how minimal acyclic complexes of finitely generated free modules arise over a commutative local ring. A standard construction gives that every totally reflexive module yields such a complex. We show that for certain rings this construction is essentially the only method of obtaining such complexes. We also give examples of rings which admit minimal acyclic complexes of finitely generated free modules which cannot be obtained by means of this construction.

Introduction

Let R be a commutative local Noetherian ring with maximal ideal $\mathfrak{m}$. An acyclic complex of finitely generated free R-modules is a complex

$$\mathbf{A}: \qquad \cdots \to A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} A_{-1} \xrightarrow{d_{-1}} A_{-2} \to \cdots$$

with A_i finitely generated and free for each i, and $\mathrm{H}(\mathbf{A}) = 0$. A complex $\mathbf{A}$ as above is said to be *totally acyclic* or a *complete resolution* whenever $\mathrm{H}(\mathbf{A}^*) = 0$, where $\mathbf{A}^* = \mathrm{Hom}_R(\mathbf{A}, R)$. Complete resolutions are important in the study of maximal Cohen-Macaulay modules, and are used in defining Tate cohomology, cf. Avramov and Martsinkovsky [6] for a recent account and related problems. The properties and uses of complete resolutions have been studied extensively. More recently, the failure of acyclic complexes of free modules to be totally acyclic is studied by Jorgensen and Şega in [14], and, in a more general setting, by Iyengar and Krause in [13]. Structure theorems for acyclic complexes of finitely generated free modules and the rings which admit non-trivial such complexes are given by Christensen and Veliche [11], in the case $\mathfrak{m}^3 = 0$.

In this paper we consider the fundamental question of how acyclic complexes of finitely generated free modules arise. In order to ignore trivial instances, we restrict our attention to *minimal* complexes $\mathbf{A}$, which are defined by the property that $d_i(A_i) \subseteq \mathfrak{m}A_{i-1}$ for all i. A standard method of obtaining minimal acyclic complexes of finitely generated free R-modules is through a process of dualization, described in Construction 2.2 below. The main objective of this paper is to demonstrate that although minimal acyclic complexes of finitely generated free modules often do arise by means of 2.2, this is not always the case, particularly when certain ring invariants are not too small.

For the purposes of this paper, we refer to complexes arising via Construction 2.2 as *sesqui-acyclic* complexes. These complexes are precisely those acyclic complexes

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of finitely generated free modules $\mathbf{A}$ satisfying $\mathrm{H}_i(\mathbf{A}^*) = 0$ for all $i \gg 0$ (see Lemma 2.3). Totally acyclic complexes are thus sesqui-acyclic, and the examples in [14] show that the converse does not hold. As noted in [11], it was previously not known whether every minimal acyclic complex of finitely generated free modules is sesqui-acyclic. The main examples of this paper show this is indeed not the case; they are given over local rings $(R, \mathfrak{m})$ satisfying $\mathfrak{m}^3 = 0$, and thus complement the results of [11].

In Section 1 we identify classes of rings with the property that every minimal acyclic complex of finitely generated free modules is totally acyclic. It is well-known that Gorenstein rings are such. We prove, for example, an extension of this result for quotients of Golod rings by Gorenstein ideals.

Recall that the *generalized Loewy length* of a local ring $(R, \mathfrak{m})$ is the integer $\ell\ell(R) = \min\{n \geq 0 \mid \mathfrak{m}^n \subseteq (\boldsymbol{x}) \text{ for some system of parameters } \boldsymbol{x} \text{ of } R\}$ and the *codimension* of R, denoted $\operatorname{codim} R$, is the number $\operatorname{edim} R - \dim R$, where $\operatorname{edim} R$ denotes the minimal number of generators of $\mathfrak{m}$. As a corollary of the result above, we show that every minimal acyclic complex of finitely generated free modules is totally acyclic whenever R is Cohen-Macaulay, and one of the following conditions holds: (i) $\operatorname{codim} R \leq 2$, (ii) $\ell\ell(R) \leq 2$, or (iii) $\operatorname{codim} R = \ell\ell(R) = 3$.

In Section 2 we give examples of minimal acyclic complexes of finitely generated free modules over Cohen-Macaulay rings R with $\operatorname{codim} R \geq 5$ and $\ell\ell(R) \geq 3$ which are not sesqui-acyclic. More precisely, we construct minimal acyclic complexes $\mathbf{A}$ of finitely generated free R-modules with the property that $\mathrm{H}_i(\mathbf{A}^*) \neq 0$ for all i. The question whether every minimal acyclic complex of finitely generated free modules is sesqui-acyclic remains open for local rings with $\operatorname{codim} R = 4$ and $\ell\ell(R) \geq 3$.

1. Growth of Betti numbers and acyclic complexes

Throughout this section, R denotes a commutative local Noetherian ring with maximal ideal $\mathfrak{m}$ and residue field k.

1.1. **Complexes.** We consider complexes of finitely generated free R-modules

$$\mathbf{A}: \qquad \cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}^{\mathbf{A}}} A_n \xrightarrow{d_n^{\mathbf{A}}} A_{n-1} \longrightarrow \cdots$$

The *trivial complex* is the complex with $A_i = 0$ for all i.

The complex $\mathbf{A}$ is said to be *acyclic* if $\mathrm{H}(\mathbf{A}) = 0$. We let $(\ )^*$ denote the functor $\operatorname{Hom}_R(\ , R)$. The *dual complex* of $\mathbf{A}$ is the complex $\mathbf{A}^*$, which has component $(A_{-n})^*$ in degree n, and differentials $d_n^{\mathbf{A}^*} = (d_{-n+1}^{\mathbf{A}})^*$. An acyclic complex of finitely generated free modules $\mathbf{A}$ is said to be *totally acyclic* if $\mathrm{H}(\mathbf{A}^*) = 0$.

The complex $\mathbf{A}$ is said to be *minimal* if $d_i^{\mathbf{A}}(A_i) \subseteq \mathfrak{m}A_{i-1}$ for all $i \in \mathbb{Z}$ (cf. [6, 8.1]).

For each integer n, the nth *syzygy module* of $\mathbf{A}$ is $\Omega^n\mathbf{A} = \operatorname{Coker} d_{n+1}^{\mathbf{A}}$.

In this section we identify several classes of rings which satisfy the property:

$$(\mathbf{a} = \mathbf{ta}) \qquad \begin{cases} \text{Every minimal acyclic complex of finitely generated free} \\ R\text{-modules is totally acyclic.} \end{cases}$$

1.2. **G-dimension.** We recall the notion of G-dimension, introduced by Auslander and Bridger [1]. A finitely generated R-module M is said to be *totally reflexive* (or, equivalently, to have *G-dimension zero*) if the following conditions hold:

(1) the natural evaluation map $M \to M^{**}$ is an isomorphism;
(2) $\operatorname{Ext}^i_R(M,R) = 0$ for all $i > 0$;
(3) $\operatorname{Ext}^i_R(M^*,R) = 0$ for all $i > 0$.

A finitely generated R-module M is said to have *Gorenstein dimension* g, denoted $\operatorname{G-dim}_R M = g$, if g is the smallest integer such that there exists an exact sequence $0 \to G_g \to \cdots \to G_1 \to G_0 \to M \to 0$ with each G_i totally reflexive. If no such integer exists, then $\operatorname{G-dim}_R M = \infty$. The Auslander-Bridger formula for G-dimension states that if $\operatorname{G-dim}_R M < \infty$, then $\operatorname{G-dim}_R M = \operatorname{depth} R - \operatorname{depth} M$.

It is clear from the definitions that an R-module M is totally reflexive if and only if $M \cong \Omega^0\mathbf{A}$ for some totally acyclic complex of finitely generated free modules $\mathbf{A}$. We collect below some variations on this fact, as suited for our purposes.

1.3. Lemma. *Let $\mathbf{A}$ be an acyclic complex of finitely generated free R-modules. The following are then equivalent.*

(1) *$\mathbf{A}$ is totally acyclic.*
(2) *$\Omega^i\mathbf{A}$ is totally reflexive for all $i \in \mathbb{Z}$.*
(3) *$\operatorname{G-dim}_R(\Omega^i\mathbf{A}) < \infty$ for some $i \in \mathbb{Z}$.*

Proof. (1) $\iff$ (2): See Avramov and Martsinkovsky [6, Lem. 2.4].

Clearly, (2) $\implies$ (3). To show (3) $\implies$ (2), note that in a short exact sequence, if two modules have finite G-dimension then so does the third (see Auslander and Bridger [1, 3.11]). A recursive use of the short exact sequences:

$$0 \to \Omega^j\mathbf{A} \to A_j \to \Omega^{j-1}\mathbf{A} \to 0$$

gives then $\operatorname{G-dim}_R(\Omega^i\mathbf{A}) < \infty$ for all i. The Auslander-Bridger formula and a counting of depths (by using the 'depth lemma' [7, 1.2.9]) along the short exact sequences above then give $\operatorname{G-dim}_R(\Omega^i\mathbf{A}) = 0$ for all i.

1.4. **Gorenstein rings.** If R is Gorenstein, then any finitely generated R-module M satisfies $\operatorname{G-dim}_R M < \infty$ (see [1]). In consequence, Lemma 1.3 shows that every Gorenstein ring R satisfies ($\mathbf{a} = \mathbf{ta}$). More generally, for R not necessarily local, Iyengar and Krause [13] show that R is Gorenstein if and only if every acyclic complex of (not necessarily finitely generated) projective modules is totally acyclic.

In contrast to the result of [13], Gorenstein rings are not the only ones that are known to satisfy ($\mathbf{a} = \mathbf{ta}$). For example, rings R with $\mathfrak{m}^2 = 0$ are known to satisfy the property that every minimal acyclic complex of finitely generated free R-modules is trivial. In Proposition 1.6 we generalize this result to a larger class of rings, characterized by a growth property for Betti numbers.

1.5. **Betti numbers.** For every finitely generate R-module M we denote by $\beta_i^R(M)$ the ith *Betti number of* M, defined to be the integer $\dim_k \operatorname{Tor}_i^R(M,k)$; it is equal to the rank of the ith free module in a minimal free resolution of M. If $\mathbf{A}$ is a minimal acyclic complex of finitely generated free R-modules, then we have:

$$\beta^R_{j-i}(\Omega^i\mathbf{A}) = \operatorname{rank}_R A_j \quad \text{for all } j \text{ and } i \text{ with } j \geq i \tag{1.5.1}$$

Moreover, the following statements are equivalent:

(1) $\mathbf{A}$ is trivial
(2) $A_i = 0$ for some $i \in \mathbb{Z}$.
(3) $\operatorname{pd}_R \Omega^n\mathbf{A} < \infty$ for some $n \in \mathbb{Z}$.

Indeed, that (1) $\implies$ (2) $\implies$ (3) is clear. To justify (3) $\implies$ (1), if $\operatorname{pd}_R \Omega^n \mathbf{A} < \infty$ for some n, then $\operatorname{pd}_R \Omega^i \mathbf{A} < \infty$ for all $i < n$. The Auslander-Buchsbaum formula for projective dimension and a count of depths gives then $\operatorname{pd}_R \Omega^i \mathbf{A} = 0$, hence $\Omega^i \mathbf{A}$ is free, for all $i < n$. Using minimality, it follows that $A_i = 0$ for all $i < n$. Minimality again shows that $A_i = 0$ for all $i \geq n$.

Many local rings share the following growth property (see Remark 1.8 below):

$$(\sharp) \quad \begin{cases} \text{There exists a positive integer } d \text{ such that for every finitely} \\ \text{generated } R\text{-module } M \text{ of infinite projective dimension there exists} \\ \text{a strictly increasing subsequence } \{\beta^R_{n_i}(M)\}_{i\geq 0} \text{ of } \{\beta^R_i(M)\}_{i\geq 0} \text{ with} \\ \text{and } id \leq n_i < (i+1)d \text{ for all } i \geq 0. \end{cases}$$

1.6. Proposition. *If the local ring R satisfies $(\sharp)$, then every minimal acyclic complex of finitely generated free R-modules is trivial.*

Proof. Let $\mathbf{A}$ be a minimal acyclic complex, and set

$$m = \max\{\operatorname{rank}_R A_i \mid 0 \leq i < d\},$$

where d is as in the definition of $(\sharp)$. Define $M = \Omega^{-md}\mathbf{A}$. If $\operatorname{pd}_R M = \infty$, then there exists a subsequence $\{\beta_{n_i}(M)\}$ of $\{\beta_n(M)\}$ such that $\beta_{n_i}(M) \geq \beta_{n_0}(M) + i$, and $id \leq n_i < (i+1)d$ for all $i \geq 0$. In particular, $\beta_{n_m}(M) \geq \beta_{n_0}(M) + m$ with $md \leq n_m < (m+1)d$. In other words,

$$\operatorname{rank}_R A_{n_m - md} \geq \operatorname{rank}_R A_{n_0 - md} + m$$

with $md \leq n_m < (m+1)d$.

Since $\operatorname{rank}_R A_{n_0 - md} > 0$, we see that $\operatorname{rank}_R A_{n_m - md} > m$. But this contradicts the definition of m, as $0 \leq n_m - md < d$. Therefore it must be that $\operatorname{pd}_R M < \infty$, hence $\mathbf{A}$ is trivial (see (1.5)).

Recall that an R-module M is said to be *perfect* if $\operatorname{pd}_R M = \operatorname{grade}_R M$. An ideal I of R is said to be *perfect* if the R-module R/I is perfect. The ideal I is said to be *Gorenstein* if it is perfect and $\beta^R_g(R/I) = 1$ for $g = \operatorname{grade}_R I$.

Since regular local rings satisfy property $(\sharp)$ vacuously, the following theorem is a generalization of the fact mentioned in 1.4, that Gorenstein rings satisfy $(\mathbf{a} = \mathbf{ta})$.

1.7. Theorem. *Assume R satisfies property $(\sharp)$. Let I be a Gorenstein ideal of R and set $S = R/I$. The local ring S satisfies then $(\mathbf{a} = \mathbf{ta})$.*

The theorem will be proved at the end of the section. We proceed now to give some applications.

1.8. *Remark.* It is known that R satisfies $(\sharp)$ in all of the following cases:

(a) R is a regular local ring.
(b) R is a Golod ring which is not a hypersurface. (Peeva, [18, Proposition 5]).
(c) $R = S/\mathfrak{n}J$, where $(S, \mathfrak{n})$ is a local ring and J is an ideal of S such that $\operatorname{rank}_k J/\mathfrak{n}J \geq 2$ (Lescot, [15, Theorem A.2]).
(d) $R = S/\mathfrak{n}J$, where $(S, \mathfrak{n})$ is a local ring and J is an $\mathfrak{n}$-primary ideal of S (Lescot, [15, Theorem A.1]).
(e) $R = S/I$, where $(S, \mathfrak{n})$ is a local ring and I is an ideal of S such that $\operatorname{rank}_k(\mathfrak{n}J/\mathfrak{n}I) \geq 2$, where $J = (I :_S \mathfrak{n})$ (Choi, [9, Theorem 1.1]).
(f) $\mathfrak{m}^3 = 0$, and either $e := \operatorname{rank}_k \mathfrak{m}/\mathfrak{m}^2 < \operatorname{rank}_k \mathfrak{m}^2$, or $P^R_k(t) \neq (1 - et + (e-1)t^2)^{-1}$ (Lescot, [15, Theorem B(1)]).

In view of Remark 1.8, Theorem 1.7 identifies many classes of rings which satisfy ($\mathbf{a} = \mathbf{ta}$). We can easily extend these classes even further by means of usual homological constructions.

1.9. PROPOSITION. *If $\mathbf{A}$ is a complex of finitely generated free R-modules, $R \to S$ is a homomorphism of local rings, and $\boldsymbol{x}$ is an R-regular sequence, then the following hold:*

(1) *$\mathbf{A}$ is acyclic (respectively, totally acyclic) if and only if $\mathbf{A} \otimes_R R/(\boldsymbol{x})$ is acyclic (respectively, totally acyclic).*
(2) *If $R \to S$ is a faithfully flat, then $\mathbf{A}$ is acyclic (respectively, totally acyclic) if and only if $\mathbf{A} \otimes_R S$ is acyclic (respectively, totally acyclic).*

PROOF. It suffices to show that $\mathbf{A}$ is acyclic if and only if $\mathbf{A} \otimes_R R/(\boldsymbol{x})$ and $\mathbf{A} \otimes_R S$ are acyclic. The statements for total acyclicity follow in view of the isomorphisms of complexes

$$\mathrm{Hom}_{R/(\boldsymbol{x})}(\mathbf{A} \otimes_R R/(\boldsymbol{x}), R/(\boldsymbol{x})) \cong \mathbf{A}^* \otimes_R R/(\boldsymbol{x}),$$

and

$$\mathrm{Hom}_S(\mathbf{A} \otimes_R S, S) \cong \mathbf{A}^* \otimes_R S.$$

Statement (2) is obvious. To prove (1), we may assume that $\boldsymbol{x}$ consists of a single regular element x. Note that there exists an exact sequence of complexes $0 \to \mathbf{A} \xrightarrow{x} \mathbf{A} \to \mathbf{A} \otimes_R R/(x) \to 0$ which gives rise in homology to the exact sequence

$$\cdots \to \mathrm{H}_i(\mathbf{A}) \xrightarrow{x} \mathrm{H}_i(\mathbf{A}) \to \mathrm{H}_i(\mathbf{A} \otimes_R R/(x)) \to \mathrm{H}_{i-1}(\mathbf{A}) \xrightarrow{x} \cdots$$

Obviously $\mathrm{H}_i(\mathbf{A}) = 0 = \mathrm{H}_{i-1}(\mathbf{A})$ implies $\mathrm{H}_i(\mathbf{A} \otimes_R R/(x)) = 0$. Also, since $\mathrm{H}_i(\mathbf{A})$ is finitely generated for each i, Nakayama's lemma gives that if $\mathrm{H}_i(\mathbf{A} \otimes_R R/(x)) = 0$, then $\mathrm{H}_i(\mathbf{A}) = 0$.

The following is an easy corollary of Proposition 1.9.

1.10. COROLLARY. *The following hold for a local ring R:*

(1) *If $\boldsymbol{x}$ is a regular R-sequence and $R/\boldsymbol{x}R$ satisfies ($\mathbf{a} = \mathbf{ta}$), then R satisfies ($\mathbf{a} = \mathbf{ta}$).*
(2) *If there exists a flat homomorphism $R \to S$ of local rings, and S satisfies ($\mathbf{a} = \mathbf{ta}$), then R satisfies ($\mathbf{a} = \mathbf{ta}$).*

We have the following corollary of Proposition 1.6/Theorem 1.7, which indicates to what extent the examples of Section 3 are minimal with respect to the invariants Loewy length and codimension.

1.11. COROLLARY. *Let R be a local Cohen-Macaulay ring of codimension c and generalized Loewy length ℓ. Then R satisfies ($\mathbf{a} = \mathbf{ta}$), provided one of the following conditions holds:*

(1) $c \le 2$
(2) $\ell \le 2$
(3) $c = \mathrm{mult}(R) - 1$.
(4) $c = \ell = 3$

PROOF. In view of Remark 1.4, we may assume that the ring R is not Gorenstein.

Under the assumption in (1), Scheja [17] proves that the ring R is either a complete intersection, or a Golod ring. Since we assumed R is not Gorenstein, R is thus Golod. The result follows from Proposition 1.6 and Peeva's result mentioned in Remark 1.8(b).

(2) In this case there exists a regular sequence $\boldsymbol{x}$ such that the square of the maximal ideal of $R/\boldsymbol{x}R$ is zero. A local ring for which the square of the maximal ideal is zero is Golod. Since the elements of the regular sequence can assumed to be in $\mathfrak{m}-\mathfrak{m}^2$ it follows from [3, 5.2.4(2)] that R is also Golod. The result thus follows from Proposition 1.6 and 1.8(b).

(3) In this case R has minimal multiplicity, and is therefore Golod (see [3, 5.2.8]). Thus again the result follows from Proposition 1.6 and 1.8(b).

(4) By hypothesis there exists a maximal R-sequence $\boldsymbol{y}$ such that the ring $S = R/\boldsymbol{y}R$ satisfies $\mathfrak{n}^3 = 0$, where $\mathfrak{n}$ is the maximal ideal of S. Theorem A of [11] gives that the Poincaré series $P^S_k(t) = \sum \beta^S_i(k)t^i$ of k over S satisfies $P^S_k(t) = (1-3t+2t^2)^{-1}$, and thus has a pole of order 1 at $t=1$. A result of Avramov [2, Theorem 3.1] shows that the ring S has an embedded deformation, that is, there exists a local ring $(R', \mathfrak{m}')$ and a regular element $x \in (\mathfrak{m}')^2$ such that $S \cong R'/xR'$. Furthermore, we have $\operatorname{codim} R' = 2$. Since we assumed that R is not Gorenstein, neither are S or R', hence R' is a Golod ring, as in the proof of part (1). Note that the ideal xR' is Gorenstein, since x is a regular element. The ring S satisfies then ($\mathbf{a} = \mathbf{ta}$) by Theorem 1.7 and Remark 1.8(b). Corollary 1.10(1) gives then the conclusion for R.

We now prepare for the proof of Theorem 1.7. For the convenience of the reader, we provide a proof of the following well-known inequality:

1.12. Lemma. *Let $I \subseteq R$ be an ideal. Set $S = R/I$ and let M be a finitely generated S-module. The following inequality holds for all $n \geq 0$:*

$$\beta^R_n(M) \leq \sum_{p+q=n} \beta^R_q(S)\beta^S_p(M)$$

Proof. Consider the change of rings spectral sequence (see [8, XVI §5])

$$\mathrm{E}^2_{p,q} = \operatorname{Tor}^S_p\left(M, \operatorname{Tor}^R_q(S,k)\right) \Longrightarrow \operatorname{Tor}^R_{p+q}(M,k)$$

Note that $\mathrm{E}^{r+1}_{p,q}$ is a subquotient of $\mathrm{E}^r_{p,q}$ for all $r \geq 2$ and the spaces $\mathrm{E}^\infty_{p,q}$ are the subfactors of a filtration of $\operatorname{Tor}^R_{p+q}(M,k)$. We have thus (in)equalities:

$$\begin{aligned}\operatorname{rank}_k \operatorname{Tor}^R_n(M,k) &= \sum_{p+q=n} \operatorname{rank}_k \mathrm{E}^\infty_{p,q} \leq \sum_{p+q=n} \operatorname{rank}_k \mathrm{E}^2_{p,q} = \\ &= \sum_{p+q=n} \beta^R_q(S)\beta^S_p(M)\end{aligned}$$

Proof of Theorem 1.7. Set $g = \operatorname{grade} I$, and let $\mathbf{A}$ be a non-trivial minimal acyclic complex of finitely generated free S-modules. Set

$$m = \max_{0 \leqslant j < d}\left\{\sum_{i=0}^{g} \beta^R_i(S)\operatorname{rank}_S A_{j-i}\right\},$$

where the integer d is defined by($\sharp$), and $M = \Omega^{-md}\mathbf{A}$. If M has infinite projective dimension over R, then property ($\sharp$) gives a subsequence $\{\beta^R_{n_i}(M)\}$ of $\{\beta^R_i(M)\}$ such that $\beta^R_{n_m}(M) \geq \beta^R_{n_0}(M) + m$ and $md \leq n_m \leq (m+1)d$. Now set $j = n_m - md$. Then $0 \leq j < d$, and by the definition of m we have

$$\beta^R_{n_m}(M) \geq \beta^R_{n_0}(M) + \sum_{i=0}^{g} \beta^R_i(S) \operatorname{rank}_S A_{j-i}\,.$$

Since $\beta^R_{n_0}(M) > 0$ and $\operatorname{rank}_S A_t = \beta^S_{t+md}(M)$ for all $t \geq -md$, in particular $\operatorname{rank}_S A_{j-i} = \beta^S_{n_m-i}(M)$ for $0 \leq i \leq g$ (since $g \leq m \leq n_m$), we obtain

$$\beta^R_{n_m}(M) > \sum_{i=0}^{g} \beta^R_i(S)\beta^S_{n_m-i}(M)$$

and this contradicts the formula given by Lemma 1.12. In consequence, we have that $\operatorname{pd}_R M < \infty$. We have thus $\text{G-dim}_R M < \infty$ and hence $\text{G-dim}_S M < \infty$, by [12, Prop. 5] (cf. also [4, 7.11]). Using Lemma 1.3, this implies that $\mathbf{A}$ is totally acyclic.

2. Sesqui-acyclic Complexes of Free Modules and Examples

Unless otherwise specified, R denotes a commutative local Noetherian ring with maximal ideal $\mathfrak{m}$ and residue field k. We use the terminology and notation of Section 1 for complexes.

Definition. We say that a complex $\mathbf{A}$ is *sesqui-acyclic* if it is acyclic and there exists an integer c such that $\mathrm{H}_i(\mathbf{A}^*) = 0$ for all $i > c$.

2.1. **Comparison of acyclic, sesqui-acyclic and acyclic complexes.** Clearly, every totally acyclic complex is sesqui-acyclic, but the converse does not hold. Indeed, Jorgensen and Şega in [14] construct minimal acyclic complexes of finitely generated free R-modules $\mathbf{C}$ with the property that $\mathrm{H}_i(\mathbf{C}^*) = 0$ if and only if $i \geq 1$. We thus have the following diagram of implications for complexes of finitely generated free R-modules (with the right implication also following directly from the definitions):

(2.1.1) $$\text{totally acyclic} \;\underset{\not\Longleftarrow}{\Longrightarrow}\; \text{sesqui-acyclic} \Longrightarrow \text{acyclic}$$

In this section we first give motivation for the definition of sesqui-acyclic complex, and then we show that the right-hand implication in (2.1.1) is not reversible. More precisely, we construct minimal acyclic complexes $\mathbf{A}$ of finitely generated free modules over codimension five local rings R with $\mathfrak{m}^3 = 0$ such that $\mathrm{H}_i(\mathbf{A}^*) \neq 0$ for all i. We then extend these to such examples over Cohen-Macaulay rings R where any choice of $\operatorname{codim} R \geq 5$ and $\ell\ell(R) \geq \operatorname{codim} R - 2$ is allowed. Note that Corollary 1.11 shows that certain restrictions on the codimension and generalized Loewy length are necessary, hence our examples are minimal, at least with respect to generalized Loewy length. We do not know whether such examples can be constructed when $\operatorname{codim} R = 4$.

As seen in Proposition 1.6, non-trivial minimal acyclic complexes of finitely generated free R-modules may not exist. When they do exist, we want to understand how they arise. A standard construction is described below (cf. [11, 3.2]).

2.2. Construction. Suppose that M is an R-module satisfying

(2.2.1) $$\operatorname{Ext}^i_R(M,R) = 0 \text{ for all } i > 0.$$

Let $\mathbf{P} \xrightarrow{\pi} M^*$ be a free resolution of M^*, with $\mathbf{P} : \cdots \longrightarrow P_2 \xrightarrow{d_2^{\mathbf{P}}} P_1 \xrightarrow{d_1^{\mathbf{P}}} P_0 \to 0$, and $\mathbf{Q} \xrightarrow{\eta} M$ be a free resolution of M, with $\mathbf{Q} : \cdots \longrightarrow Q_2 \xrightarrow{d_2^{\mathbf{Q}}} Q_1 \xrightarrow{d_1^{\mathbf{Q}}} Q_0 \to 0$. By condition (2.2.1), the complex

$$0 \longrightarrow M^* \xrightarrow{\eta^*} Q_0^* \xrightarrow{(d_1^{\mathbf{Q}})^*} Q_1^* \xrightarrow{(d_2^{\mathbf{Q}})^*} Q_2^* \longrightarrow \cdots$$

is exact, and thus one can splice the complexes $\mathbf{P}$ and $\mathbf{Q}^* = \operatorname{Hom}_R(\mathbf{Q}, R)$ together to obtain an acyclic complex of free modules:

$$\mathbf{P}|\mathbf{Q}^* : \quad \cdots \longrightarrow P_2 \xrightarrow{d_2^{\mathbf{P}}} P_1 \xrightarrow{d_1^{\mathbf{P}}} P_0 \xrightarrow{\eta^* \circ \pi} Q_0^* \xrightarrow{(d_1^{\mathbf{Q}})^*} Q_1^* \xrightarrow{(d_2^{\mathbf{Q}})^*} Q_2^* \longrightarrow \cdots$$

with the convention that $(\mathbf{P}|\mathbf{Q}^*)_i = P_i$ for $i \geq 0$, and $(\mathbf{P}|\mathbf{Q}^*)_i = Q^*_{-i-1}$ for $i < 0$. This complex is minimal whenever $\mathbf{P}$ and $\mathbf{Q}$ are chosen minimal and M has no non-zero free direct summand. It is non-trivial if M is non-zero.

Recall that the nth *shift* of the complex $\mathbf{A}$ is the complex $\Sigma^n \mathbf{A}$ with $(\Sigma^n \mathbf{A})_i = A_{i-n}$ and $d_i^{\Sigma^n \mathbf{A}} = (-1)^n d_{i-n}^{\mathbf{A}}$. We write $\mathbf{A}_{\geqslant n}$ for the complex with ith component (respectively, differential) equal to A_i (respectively, $d_i^{\mathbf{A}}$) if $i \geq n$ (respectively, $i > n$) and 0 if $i < n$ (respectively, $i \leq n$).

2.3. Lemma. *Let $\mathbf{A}$ be an acyclic complex of finitely generated free R-modules. The following are equivalent:*

1. *$\mathbf{A}$ is sesqui-acyclic.*
2. *There exist an integer s and an R-module M satisfying* (2.2.1) *such that $\mathbf{A}$ is isomorphic to $\Sigma^s(\mathbf{P}|\mathbf{Q}^*)$, where the complex $\mathbf{P}|\mathbf{Q}^*$ is defined as in Construction* 2.2*, with $\mathbf{P}$ a free resolution of M^* and $\mathbf{Q}$ a free resolution of M.*

Proof. (1) $\Longrightarrow$ (2). Assume that $\mathbf{A}$ satisfies $\mathrm{H}_i(\mathbf{A}^*) = 0$ for all $i > c$. Let s be any integer satisfying $s \geq c$ and set $M = \Omega^s(\mathbf{A}^*)$. Then $\mathbf{Q} = \Sigma^{-s}((\mathbf{A}^*)_{\geqslant s})$ is a free resolution of M. Since $\mathbf{A}$ is exact, we have $\mathrm{H}_i(\mathbf{Q}^*) = 0$ for all $i < 0$, hence $\operatorname{Ext}^i_R(M, R) = 0$ for all $i > 0$. Note that we have $M^* = \Omega^{-s+1}\mathbf{A}$, and hence $\mathbf{P} = \Sigma^{s-1}(\mathbf{A}_{\geqslant -s+1})$ is a free resolution of M^*, and thus $\mathbf{A} \cong \Sigma^{-s+1}(\mathbf{P}|\mathbf{Q}^*)$.

(2) $\Longrightarrow$ (1). Suppose that $\mathbf{A} \cong \Sigma^s(\mathbf{P}|\mathbf{Q}^*)$ for some s, with $\mathbf{P}|\mathbf{Q}^*$ as in 2.2. One has

$$\mathrm{H}_i(\mathbf{A}^*) \cong \mathrm{H}_{i+s-1}(\mathbf{Q}^{**}) \cong \mathrm{H}_{i+s-1}(\mathbf{Q}) = 0 \quad \text{for all } i > -s$$

Lemma 2.3 thus shows that sesqui-acyclic complexes are precisely those acyclic complexes which come about through a process of dualization. In [11, 3.2], Christensen and Veliche noted that all known examples of minimal acyclic complexes of finitely generated free R-modules arise by means of this process of 2.2. The authors also raised the question in [11, 3.4] of whether this is always the case, or in terms of our discussion, whether the right-hand implication in (2.1.1) is reversible. We now show that it is not reversible.

From now on, the ring R and the complex $\mathbf{A}$ are as defined below.

2.4. **The main example.** Let k be a field and $\alpha \in k$ be non-zero. Consider the quotient ring

$$R = k[X_1, X_2, X_3, X_4, X_5]/I$$

where the X_i are indeterminates (each of degree one), and I is the ideal generated by the following 11 homogeneous quadratic relations:

$$X_1^2,\ X_4^2,\ X_2X_3,\ \alpha X_1X_2\ + X_2X_4,\ X_1X_3 + X_3X_4,$$
$$X_2^2,\ X_2X_5 - X_1X_3,\ X_3^2 - X_1X_5,\ X_4X_5,\ X_5^2, X_3X_5$$

As a vector space over k, R has a basis consisting of the following 10 elements:

$$1,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_1x_2,\ x_1x_3,\ x_1x_4,\ x_1x_5$$

where x_i denote the residue classes of X_i modulo I. Since I is generated by homogeneous elements, R is graded, and has Hilbert series $1+5t+4t^2$. Moreover R has codimension five, and it is local with maximal ideal $\mathfrak{m} = (x_1, \dots, x_5)$ satisfying $\mathfrak{m}^3 = 0$.

For each integer $i \in \mathbb{Z}$ we let $d_i \colon R^2 \to R^2$ denote the map given with respect to the standard basis of R^2 by the matrix

$$\begin{pmatrix} x_1 & \alpha^i x_2 \\ x_3 & x_4 \end{pmatrix}$$

Consider the sequence of homomorphisms:

$$\mathbf{A}: \quad \cdots \to R^2 \xrightarrow{d_{i+1}} R^2 \xrightarrow{d_i} R^2 \xrightarrow{d_{i-1}} R^2 \to \cdots$$

2.5. Theorem. *The sequence* $\mathbf{A}$ *is a minimal acyclic complex of free* R*-modules with* $\mathrm{H}_i(\mathbf{A}^*) \neq 0$ *for all* $i \in \mathbb{Z}$.

We have thus:

2.6. Corollary. *The minimal acyclic complex* $\mathbf{A}$ *is not sesqui-acyclic.*

2.7. *Remark.* When $\alpha \in k$ is an element of infinite multiplicative order, the complex $\mathbf{A}$ is non-periodic. When α has multiplicative order s for some integer $s > 0$, one has that $\mathbf{A}$ is periodic of period s.

2.8. *Remark.* Christensen and Veliche ask in [11, 3.5] whether every acyclic complex of free modules $\mathbf{C}$ with $\{\operatorname{rank} \mathbf{C}_i\}$ constant, over a local ring with $\mathfrak{m}^3 = 0$, is totally acyclic. Theorem 2.5 gives a negative answer to this question.

Proof of Theorem 2.5. Using the defining relations of R, one can easily show that $d_id_{i+1} = 0$ for all i, hence $\mathbf{A}$ is a complex.

We let (a, b) denote an element of R^2 written in the standard basis of R^2 as a free R-module. For each i, the k-vector space $\operatorname{Im} d_i$ is generated by the elements:

$$\begin{aligned}
d_i(1,0) &= (x_1, x_3) & d_i(x_5,0) &= (x_1x_5, 0)\\
d_i(0,1) &= (\alpha^i x_2, x_4) & d_i(0,x_1) &= (\alpha^i x_1x_2, x_1x_4)\\
d_i(x_1,0) &= (0, x_1x_3) & d_i(0,x_2) &= (0, -\alpha x_1x_2)\\
d_i(x_2,0) &= (x_1x_2, 0) & d_i(0,x_3) &= (0, -x_1x_3)\\
d_i(x_3,0) &= (x_1x_3, x_1x_5) & d_i(0,x_4) &= (-\alpha^{i+1} x_1x_2, 0)\\
d_i(x_4,0) &= (x_1x_4, -x_1x_3) & d_i(0,x_5) &= (\alpha^i x_1x_3, 0)
\end{aligned}$$

Excluding $d_i(0, x_3)$ and $d_i(0, x_4)$, the above equations provide 10 linearly independent elements in $\operatorname{Im} d_i$. Thus $\operatorname{rank}_k(\operatorname{Im} d_i) = 10$ for all i. Since

$$\operatorname{rank}_k \operatorname{Ker} d_i + \operatorname{rank}_k \operatorname{Im} d_i = \operatorname{rank}_k R^2 = 20$$

we have $\dim \operatorname{Ker} d_i = 10$ for all i. Thus, $\operatorname{Im} d_{i+1} = \operatorname{Ker} d_i$ for all i, so that $\mathbf{A}$ is acyclic.

To prove $\mathrm{H}_i(\mathbf{A}^*) \neq 0$, we have that $d_i^* = (d_i)^* \colon R^2 \to R^2$ is represented with respect to the standard basis of R^2 by the matrix

$$\begin{pmatrix} x_1 & x_3 \\ \alpha^i x_2 & x_4 \end{pmatrix}$$

For each i, the vector space $\operatorname{Im} d_i^*$ is generated by the following elements

$$\begin{aligned} d_i^*(1,0) &= (x_1, \alpha^i x_2) & d_i^*(x_5,0) &= (x_1x_5, \alpha^i x_1x_3) \\ d_i^*(0,1) &= (x_3, x_4) & d_i^*(0,x_1) &= (x_1x_3, x_1x_4) \\ d_i^*(x_1,0) &= (0, \alpha^i x_1x_2) & d_i^*(0,x_2) &= (0, -\alpha x_1x_2) \\ d_i^*(x_2,0) &= (x_1x_2, 0) & d_i^*(0,x_3) &= (x_1x_5, -x_1x_3) \\ d_i^*(x_3,0) &= (x_1x_3, 0) & d_i^*(0,x_4) &= (-x_1x_3, 0) \\ d_i^*(x_4,0) &= (x_1x_4, -\alpha^{i+1}x_1x_2) & d_i^*(0,x_5) &= (0,0) \end{aligned}$$

Excluding $d_i^*(0,x_2)$, $d_i^*(0,x_4)$, and $d_i^*(0,x_5)$ which are redundant, we have only 9 linearly independent elements in $\operatorname{Im} d_i^*$, hence $\operatorname{rank}_k \operatorname{Im} d_i^* = 9$ for every i. It follows that $\operatorname{rank}_k \operatorname{Ker} d_i^* = 11$ for all i, hence $\mathrm{H}_i(\mathbf{A}^*) \neq 0$.

One can easily get examples of minimal acyclic complexes which are not sesqui-acyclic over local rings of any codimension larger than five as follows.

Let $n \geq 1$ be an integer, and $y_1, \dots, y_n$ be indeterminates over k. Define R_n to be the local ring obtained by localizing $R \otimes_k k[y_1, \dots, y_n]$ at the maximal ideal

$$\mathfrak{m}_n = (x_i \otimes 1, 1 \otimes y_j \mid 1 \leq i \leq 5, 1 \leq j \leq n)$$

Now let $\mathbf{A}_n$ denote the sequence

$$\cdots \to R_n^2 \xrightarrow{d_{i+1}^n} R_n^2 \xrightarrow{d_i^n} R_n^2 \xrightarrow{d_{i-1}^n} R_n^2 \to \cdots$$

where d_i^n denotes the map $d_i \otimes_k k[y_1, \dots, y_n]$ localized at $\mathfrak{m}_n$.

Let $p_1, \dots, p_n$ be positive integers ≥ 2, set $\ell = \sum_{i=1}^n (p_i - 1) + 3$, and consider the R_n-sequence $\boldsymbol{y} = 1 \otimes y_1^{p_1}, \dots, 1 \otimes y_n^{p_n}$.

2.9. Corollary. *The ring R_n is a local Cohen-Macaulay ring with* $\operatorname{codim}(R_n) = 5$, $\dim(R_n) = n$ *and* $\ell\ell(R_n) = 3$, *and* $S_n = R_n/(\boldsymbol{y})$ *is an artinian local with* $\operatorname{codim}(S_n) = n+5$ *and* $\ell\ell(S_n) = \ell$, *such that the following hold:*

(1) $\mathbf{A}_n$ *is a minimal acyclic complex of free R_n-modules which is not sesqui-acyclic.*
(2) $\mathbf{A}_n \otimes_{R_n} R_n/(\boldsymbol{y})$ *is a minimal acyclic complex of free S_n-modules which is not sesqui-acyclic.*

Proof. The statements about R_n and S_n are clear.

Since $R \to R \otimes_k k[y_1, \dots, y_n]$ is a faithfully flat embedding of rings, the complex $\mathbf{A} \otimes_k k[y_1 \dots, y_n]$ stays exact, and then too after localizing. Thus $\mathbf{A}_n$ is acyclic (and obviously minimal).

We have $\mathrm{H}_i(\mathbf{A}^*) \otimes_k k[y_1, \dots, y_n] \cong \mathrm{H}_i(\mathbf{A}^* \otimes_k k[y_1, \dots, y_n])$ for all i. Moreover, $\mathbf{A}^* \otimes_k k[y_1, \dots, y_n]$ localized at $\mathfrak{m}_n$ is isomorphic to $(\mathbf{A}_n)^*$. Since $\mathrm{H}_i(\mathbf{A}^*) \neq 0$ for all i by Theorem 2.5, it follows that $\mathrm{H}_i((\mathbf{A}_n)^*) \neq 0$ for all $i \in \mathbb{Z}$. Hence $\mathbf{A}_n$ is not sesqui-acyclic. The statement about $\mathbf{A}_n \otimes_{R_n} R_n/(\boldsymbol{y})$ follows from Proposition 1.9.

In summary, we have the following diagram of implications for complexes of finitely generated free modules over a local ring R:

$$\text{totally acyclic} \quad \underset{\not\Longleftarrow}{\Longrightarrow} \quad \text{sesqui-acyclic} \quad \underset{\not\Longleftarrow}{\Longrightarrow} \quad \text{acyclic}$$

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References

1. M. Auslander, M Bridger, *Stable module theory*, Memoirs of the American Mathematical Society **94**, AMS, Providence, RI, 1969.
2. L. L. Avramov, *Homological asymptotics of modules over local rings*, Commutative Algebra; Berkeley, 1987 (M. Hochster, C. Huneke J. Sally, eds.) MSRI Publ **15**, Springer, New York 1989 Birkhuser, pp. 33–62.
3. L. L. Avramov, *Infinite free resolutions*, in: Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math. **166**, Birkhuser, Basel, 1998; pp. 1-118.
4. L. L. Avramov, H.-B. Foxby, *Ring homomorphisms and finite Gorenstein dimension*, Proc. London Math. Soc. **75** (1997), 241–270.
5. L. L. Avramov, V. N. Gasharov, and I. V. Peeva, *Complete intersection dimension*, Publ. Math. I.H.E.S. **86** (1997), 67–114.
6. L. L. Avramov, A. Martsinkovsky, *Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension*, Proc. London Math. Soc. **85** (2002), 393–440.
7. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge studies in advanced mathematics, Vol. 39, Revised edition, Cambridge University Press.
8. H. Cartan, and S. Eilenberg, *Homological algebra*, Princeton University Press, Princeton, New Jersey, 1956.
9. S. Choi, *Exponential growth of betti numbers*, J. Algebra **152** (1992), 20–29.
10. L. W. Christensen, *Gorenstein dimensions*, Lecture Notes in Mathematics, 1747. Springer-Verlag, Berlin, 2000.
11. L. W. Christensen, O. Veliche, *Acyclicity over local rings with radical cube zero*, Illinois J. Math. (to appear).
12. E. S. Golod, *G-dimension and generalized perfect ideals*, Proc. Steklov Inst. Mat. **165** (1984), 67–71.
13. S. Iyengar, H. Krause, *Acyclicity versus total acyclicity for complexes over Noetherian rings*, Documenta Math. **11** (2006), 207–240.
14. D. A. Jorgensen, L. M. Şega, *Independence of the total reflexivity conditions for modules*, Algebr. Represent. Theory **9** (2006), 217–226.
15. J. Lescot, *Asymptotic properties of Betti numbers of modules over certain rings*, J. Pure Appl. Algebra **38** (1985), 287–298.
16. H. Matsumura, *Commutative ring theory*. Cambridge University Press, 1986.
17. G. Scheja, *Uber die Bettizahlen lokaler Ringe*, Math. Ann. **155** (1964), 155–172.
18. I. Peeva, *Exponential growth of Betti numbers*, J. Pure Appl. Algebra **126** (1998), 317–323.

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