ASYMMETRIC COMPLETE RESOLUTIONS AND VANISHING OF EXT OVER GORENSTEIN RINGS

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Abstract. We construct a class of Gorenstein local rings R which admit minimal complete R-free resolutions $\boldsymbol{C}$ such that the sequence $\{\operatorname{rank}_R C_i\}$ is constant for $i < 0$, and grows exponentially for all $i > 0$. Consequently, over these rings we show that there exist finitely generated R-modules M and N such that $\operatorname{Ext}^i_R(M, N) = 0$ for all $i > 0$, but $\operatorname{Ext}^i_R(N, M) \neq 0$ for all $i > 0$.

Introduction

Let R be a commutative local Noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$, and let M,N denote finitely generated R-modules. We write $\nu_R(M)$ for the minimal number of generators of M.

It is well-known that R is Gorenstein if and only if the following remarkable symmetry is satisfied: for any module M we have $\operatorname{Ext}^i_R(M, k) = 0$ for all $i \gg 0$ if and only if $\operatorname{Ext}^i_R(k, M) = 0$ for all $i \gg 0$ (equivalently, $\operatorname{pd}_R(M) < \infty$ if and only if $\operatorname{id}_R(M) < \infty$). A fundamental question is whether this statement still holds when k is replaced with any module N. More generally, does the Gorenstein property of R translate into similarities in the asymptotic behavior of the sequences $\{\nu_R\big(\operatorname{Ext}^i_R(M, N)\big)\}_{i\geq 0}$ and $\{\nu_R\big(\operatorname{Ext}^i_R(N, M)\big)\}_{i\geq 0}$?

To describe the asymptotic behavior of such sequences, we rely on established notions of growth. We say that a sequence of positive integers $\{c_i\}_{i\geq 0}$ has *polynomial growth of degree* d if there exist polynomials $f(t)$ and $g(t)$, both of degree d and having the same leading term, such that $g(i) \leq c_i \leq f(i)$ for all $i \gg 0$. (We adopt the convention that the zero polynomial has degree -1.) We say that $\{c_i\}_{i\geq 0}$ has *exponential growth* if there exist $a, b \in \mathbb{R}$ with $1 < a \leq b$ such that $a^i \leq c_i \leq b^i$ for all $i \gg 0$. Throughout this paper, we say that two sequences have *the same growth*, or *the same asymptotic behavior* whenever the following holds: if one of them has polynomial growth, then the other one has polynomial growth of the same degree, and if one of them has exponential growth, then the other one has exponential growth as well.

It is thus meaningful to ask:

Question. *If R is a Gorenstein ring, then do the sequences $\{\nu_R\big(\operatorname{Ext}^i_R(M, N)\big)\}_{i\geq 0}$ and $\{\nu_R\big(\operatorname{Ext}^i_R(N, M)\big)\}_{i\geq 0}$ have the same growth?*

Since complete intersection rings are Gorenstein, a foundation is laid by the following theorem of Avramov and Buchweitz [2, 5.6]:

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Theorem AB. *Suppose R is a complete intersection ring. Then for any pair of finitely generated R-modules M and N the sequences $\{\nu_R\big(\operatorname{Ext}^i_R(M,N)\big)\}_{i\geq 0}$ and $\{\nu_R\big(\operatorname{Ext}^i_R(N,M)\big)\}_{i\geq 0}$ both have polynomial growth of the* same *degree.*

Complete intersection rings R are actually characterized by polynomial growth of the sequences $\{\nu_R\big(\operatorname{Ext}^i_R(M,N)\big)\}_{i\geq 0}$. Therefore, when attempting to extend Theorem AB to arbitrary Gorenstein rings, one needs to keep in mind that the sequences under consideration often have exponential growth. Although it is known that these sequences are always exponentially bounded above, it is not known whether intermediate types of growth are possible.

The answer to the question above was not previously known for arbitrary Gorenstein rings, even in the particular case when $N=k$. In this case, the notation can be simplified: the numbers $\beta_i^R(M)=\nu_R\big(\operatorname{Ext}^i_R(M,k)\big)$ are called the *Betti* numbers of M and the numbers $\mu^i_R(M)=\nu_R\big(\operatorname{Ext}^i_R(k,M)\big)$ are called the *Bass* numbers of M (associated to the maximal ideal $\mathfrak{m}$). The question becomes thus whether the Betti and the Bass sequence of M have the same asymptotic behavior.

The purpose of this paper is to give a fairly complete answer to the question above. In view of the usual reduction by a maximal regular sequence, we may assume whenever convenient that R is artinian. We first show that if $\mathfrak{m}^3=0$ or $\operatorname{codim} R\leq 4$ (where $\operatorname{codim} R$ in general denotes the number $\nu_R(\mathfrak{m})-\dim R$) then the Betti and the Bass sequence of a finitely generated R-module M have the same growth. However, the main result of the paper provides a negative answer:

Theorem. *There exist Gorenstein rings R with $\mathfrak{m}^4=0$ and $\operatorname{codim} R=6$, and finitely generated R-modules, such that their Betti sequence is constant (respectively has exponential growth) and their Bass sequence has exponential growth (respectively is constant). Moreover, there exist finitely generated R-modules M, N such that*

$$\operatorname{Ext}^i_R(M,N)=0 \quad \text{for all } i>0 \quad \text{and} \quad \operatorname{Ext}^i_R(N,M)\neq 0 \quad \text{for all } i>0\,.$$

The question above has surfaced in recent literature under various formulations, as we shall discuss next.

Symmetry in the vanishing of Ext. Theorem AB, restricted to the case of polynomial growth of degree -1, shows that any complete intersection ring R satisfies the following property:

(**ee**) If M and N are finitely generated R-modules such that $\operatorname{Ext}^i_R(M,N)=0$ for all $i\gg 0$, then $\operatorname{Ext}^i_R(N,M)=0$ for all $i\gg 0$.

The authors of [2] asked the question of whether all Gorenstein rings satisfy (**ee**). It was subsequently established in [7] and [13] that (**ee**) holds for certain classes of Gorenstein local rings $(R,\mathfrak{m})$ other than the complete intersection rings, for example Gorenstein rings with $\mathfrak{m}^3=0$, and Gorenstein rings with $\operatorname{codim} R\leq 4$.

In [7], Huneke and Jorgensen introduce a class of Gorenstein rings, called AB rings, and prove that any AB ring satisfies (**ee**). In [8], the authors constructed Gorenstein rings which are not AB, but these examples failed to disprove (**ee**).

Betti numbers versus Bass numbers; complete resolutions. In the special case when $N=k$, the Question has been previously posed in the more general context of complete resolutions. A *complete resolution* of the R-module M is a complex $\boldsymbol{C}$ of finitely generated free R-modules with differentials $d_i\colon C_i\to C_{i-1}$

such that the complexes $\boldsymbol{C}$ and $\operatorname{Hom}_R(\boldsymbol{C}, R)$ are both exact, and such that $\boldsymbol{C}_{\gg 0} = \boldsymbol{F}_{\gg 0}$ for some free resolution $\boldsymbol{F}$ of M. The complex $\boldsymbol{C}$ is said to be *minimal* if $d_i(C_i) \subseteq \mathfrak{m}C_{i-1}$ for all $i \in \mathbb{Z}$. If R is Gorenstein, then every finitely generated R-module M has a minimal complete resolution $\boldsymbol{C}$. Moreover, any two minimal complete resolutions of M are isomorphic, cf. [3, 8.4], hence the numbers $\operatorname{rank}_R C_i$ are uniquely determined. We say that $\boldsymbol{C}$ has *symmetric growth* if the sequences $\{\operatorname{rank}_R C_i\}_{i \geq 0}$ and $\{\operatorname{rank}_R C_{-i}\}_{i \geq 0}$ have the same growth.

If R is Gorenstein and $\boldsymbol{C}$ is a minimal complete resolution of a maximal Cohen-Macaulay R-module M, then $\beta_i^R(M) = \operatorname{rank}_R C_i$ and $\mu_R^{i+d-1}(M) = \operatorname{rank}_R C_{-i}$ for all $i \geq 1$, where $d = \dim R$. Therefore the initial question can be translated to the question of whether $\boldsymbol{C}$ has symmetric growth. Similar versions of the latter have been previously posed in [3, 9.2] and in [9], without the assumption that R is Gorenstein. In [9] we constructed doubly infinite minimal exact complexes of free modules which had asymmetric growth; however, the ring was not Gorenstein and these complexes were not complete resolutions.

The paper is organized as follows: In Section 1 we use results of Avramov [1] and Sun [17] to prove that any complete resolution has symmetric growth over Gorenstein rings with $\mathfrak{m}^3 = 0$ or $\operatorname{codim} R \leq 4$.

In Section 2 we prove that there exist Gorenstein rings R with $\mathfrak{m}^4 = 0$ and $\operatorname{codim} R = 6$ which admit complete resolutions $\boldsymbol{C}$ for which $\{\operatorname{rank}_R C_i\}_{i \geq 0}$ is constant (respectively, grows exponentially), and $\{\operatorname{rank}_R C_{-i}\}_{i \geq 0}$ grows exponentially (respectively, is constant). We do not know whether such asymmetric complete resolutions exist when $\operatorname{codim} R = 5$.

Using the results of Section 2, we prove in Section 3 that there exist finitely generated modules M, N which give counterexamples to (**ee**); the ring is the same as in Section 2. The module N has minimal possible length for such a counterexample, namely length 2. The results of this section are stated in terms of Tate (co)homology: we show that the Tate cohomology groups $\widehat{\operatorname{Ext}}_R^i(M, N)$ vanish for all $i > 0$, but do not vanish for all $i < 0$.

In Appendix A we establish the structure and relevant properties of the rings R from Sections 2 and 3. These rings are similar to those constructed in [6], [8], [9].

1. Symmetric growth of complete resolutions

In this section we show that there exist certain classes of Gorenstein rings, other than the class of complete intersection rings, for which all complete resolutions have symmetric growth.

Let $(R, \mathfrak{m}, k)$ be a local ring as in the introduction. If R is Gorenstein, then a complete resolution of a finitely generated R-module M has symmetric growth if and only if the Betti sequence $\{\beta_i^R(M)\}_i$ and the Bass sequence $\{\mu_R^i(M)\}_i$ have the same growth. For the convenience of the reader, we prove this in Lemma 1.2.

1.1. The asymptotic behavior of the Betti sequences remains unchanged upon passing to syzygies. When the ring is Gorenstein, the same is true for the Bass sequences. We may thus assume, whenever convenient, that M is a maximal Cohen-Macaulay module over R.

We let M^* denote the R-module $\operatorname{Hom}_R(M, R)$. If $\boldsymbol{D}$ is a complex, then $\boldsymbol{D}^*$ denotes the complex with $(\boldsymbol{D}^*)_i = (D_{-i})^*$ for each i, and with induced differentials.

1.2. Lemma. *Let R be a Gorenstein local ring of dimension d, let M be a finitely generated maximal Cohen-Macaulay R-module, and $\boldsymbol{C}$ a minimal complete resolution of M. The following equalities then hold:*

(1) $\beta_i^R(M) = \operatorname{rank}_R C_i$ *for all* $i \geq 0$, *and* $\mu_R^{i+d-1}(M) = \operatorname{rank}_R C_{-i}$ *for all* $i \geq 1$;
(2) $\beta_i^R(M^*) = \mu_R^{i+d}(M)$ *for all* $i \geq 0$.

Proof. (1) If $d = 0$, the statement is clear: $\boldsymbol{C}_{\geqslant 0}$ is a minimal free resolution of M over R, and $\boldsymbol{C}_{\leqslant -1}$ is a minimal injective resolution of M over R. If $d > 0$, then let $\boldsymbol{x} = x_1, \ldots, x_d$ be a maximal regular sequence for both R and M. Note that $\boldsymbol{C}/(\boldsymbol{x})\boldsymbol{C} = \boldsymbol{C} \otimes_R R/(\boldsymbol{x})$ is a minimal complete resolution of $\overline{M} = M/(\boldsymbol{x})M$ over the zero-dimensional Gorenstein ring $\overline{R} = R/(\boldsymbol{x})$. The conclusion then follows from the isomorphisms $\operatorname{Ext}_R^i(M,k) \cong \operatorname{Ext}_{\overline{R}}^i(\overline{M},k)$ and $\operatorname{Ext}_R^{i+d}(k,M) \cong \operatorname{Ext}_{\overline{R}}^i(k,\overline{M})$, which hold for for all $i \geq 0$ cf. for example [11, p. 140].

(2) Note that $(\boldsymbol{C}_{\leqslant -1})^*$ is a minimal free resolution of M^* over R, hence for all $i \geq 0$ we have $\beta_i^R(M^*) = \operatorname{rank}_R(C_{-i-1})^* = \operatorname{rank}_R C_{-i-1}$. By part (1), the last expression is equal $\mu_R^{i+d}(M)$. □

1.3. If R is a Gorenstein ring with $\operatorname{codim} R \geq 2$, then R has multiplicity at least $\operatorname{codim} R + 2$. Otherwise, when $\operatorname{codim} R \leq 1$, the multiplicity is at least $\operatorname{codim} R + 1$. In either case, when equality holds we say that R is *Gorenstein of minimal multiplicity.* If R is furthermore Artinian, then R has minimal multiplicity if and only if $\mathfrak{m}^3 = 0$.

1.4. If R is Gorenstein of minimal multiplicity with $\operatorname{codim} R \geq 3$, then for each finitely generated R-module M either M has finite projective dimension, or the sequence $\{\beta_i^R(M)\}_i$ has exponential growth. Indeed, if $\dim R = 0$, then $\mathfrak{m}^3 = 0$ and the result is proved by Sjödin [15], cf. also Lescot [10]. If $\dim R > 0$, then the reduction to the zero dimensional case can be done as described in [13, 1.7].

1.5. Assume now that R is Gorenstein and $\operatorname{codim} R \leq 4$. Avramov [1] and Sun [17] classified the possible behavior of the Betti numbers of a finitely generated R-module M. They show that the Betti sequence has either polynomial growth, or exponential growth. Note that Avramov and Sun use the terminology of *strong* polynomial/exponential growth for describing the same concepts that we are concerned with, only that we omit the word "strong". The classification involves the notion of *virtual projective dimension.* We recall this notion for the reader's convenience: let M be a finitely generated module over a local ring R (not necessarily Gorenstein). If the residue field k of R is infinite, set $\widetilde{R} = \widehat{R}$, the $\mathfrak{m}$-adic completion of R; if k is finite, set $\widetilde{R}$ to be the maximal-ideal-adic completion of $R[Y]_{\mathfrak{m}R[Y]}$, where Y is an indeterminate. We say that a map of local rings $\widetilde{R} \leftarrow Q$ is an *embedded deformation* of R if its kernel is generated by a Q-regular sequence contained in the square of the maximal ideal of Q. The *virtual projective dimension* of M is the number

$$\operatorname{vpd}_R M = \min\{\operatorname{pd}_Q(M \otimes_R \widetilde{R}) \mid \widetilde{R} \leftarrow Q \text{ is an embedded deformation of } \widetilde{R}\}$$

A similar invariant, called *virtual injective dimension* and denoted $\operatorname{vid}_R M$, can be defined by replacing $\operatorname{pd}_Q(M \otimes_R \widetilde{R})$ with $\operatorname{id}_Q(M \otimes_R \widetilde{R})$ in the formula above. In [1] and [17] it is shown that if R is Gorenstein with $\operatorname{codim} R \leq 4$, then the Betti numbers of M fall into one of two categories, each described by equivalent conditions:

(1) $\{\beta_i^R(M)\}_i$ has polynomial growth of degree q if and only if $\operatorname{vpd}_R M = \operatorname{depth} R - \operatorname{depth} M + q + 1$;
(2) $\{\beta_i^R(M)\}_i$ has exponential growth if and only if $\operatorname{vpd}_R M = \infty$.

Using Lemma 1.2(2), we see that the Bass numbers of M have the same behavior: they either have polynomial growth or else they have exponential growth.

We are now ready to prove the main result of this section, which is an assemblage of the results of [1] and [17] listed above.

1.6. **Theorem.** *Assume that R is Gorenstein, and either R has minimal multiplicity, or* $\operatorname{codim} R \leq 4$. *If M is a finitely generated R-module, then one of the following statements is satisfied:*

(1) *Both sequences $\{\beta_i^R(M)\}_{i\geq 0}$ and $\{\mu_R^i(M)\}_{i\geq 0}$ have polynomial growth of the same degree.*
(2) *Both sequences $\{\beta_i^R(M)\}_{i\geq 0}$ and $\{\mu_R^i(M)\}_{i\geq 0}$ have exponential growth.*

Proof. Assume first that R has minimal multiplicity. If $\operatorname{codim} R \leq 2$, then R is a complete intersection, and Theorem AB in the introduction shows that M satisfies (1). Assume now $\operatorname{codim}(R) \geq 3$. If $\operatorname{pd}_R M = \infty$, then $\operatorname{id}_R M = \infty$ as well, and it can be immediately seen from 1.2(2) and 1.4 that M satisfies condition (2). If $\operatorname{pd}_R M$ is finite, then $\operatorname{id}_R M$ is also finite, and M satisfies (1).

Assume $\operatorname{codim} R \leq 4$. Since R is Gorenstein, the same is true for any ring Q in an embedded deformation $\widetilde{R} \leftarrow Q$, and so it is clear that $\operatorname{vid}_R M = \operatorname{vpd}_R M + \operatorname{depth} M$. It follows then from [1, 1.8] and 1.5 that M satisfies (1) whenever $\operatorname{vpd}_R M$ and $\operatorname{vid}_R M$ are finite and M satisfies (2) whenever they are infinite. □

Lemma 1.2 and 1.1 now yield:

1.7. **Corollary.** *Assume R is Gorenstein. If R has minimal multiplicity, or* $\operatorname{codim} R \leq 4$, *then any minimal complete resolution over R has symmetric growth.* □

2. Asymmetric growth of complete resolutions

In this section we show that complete resolutions need not be symmetric when $\mathfrak{m}^4 = 0$ and $\operatorname{codim} R = 6$.

Let k be a field which is not algebraic over a finite field and let $\alpha \in k$ be an element of infinite multiplicative order. Throughout the whole section we consider the ring R to be defined as follows.

2.1. Let $P = k[T, U, V, X, Y, Z]$ be the polynomial ring in six variables (each of degree one) and set $R = P/I$, where I is the ideal generated by the following fifteen quadratic polynomials:

$$Z^2,\ UZ - TX - \alpha UV,\ U^2,\ YZ + VY,\ UY,\ Y^2 - TX - (\alpha - 1)UV,$$
$$XZ + \alpha VX,\ UX,\ XY,\ X^2 - TX - TV,\ TZ + TY + \alpha VX,\ TU,$$
$$TY - VX + TV,\ T^2 + (\alpha + 1)UV - VY,\ V^2.$$

Let t, u, v, x, y, z denote the residue classes of the variables modulo I, and $\mathfrak{m}$ denote the ideal they generate.

2.2. **Proposition.** *The ring R is local, with maximal ideal $\mathfrak{m}$, and satisfies the following properties:*

(1) *R has Hilbert series $H_R(t) = 1 + 6t + 6t^2 + t^3$. More precisely, a basis of R over k is given by the following fourteen elements:*

$$1,\ t,\ u,\ v,\ x,\ y,\ z,\ tv,\ uv,\ vx,\ vy,\ vz,\ tx,\ vtx$$

(2) *R is Gorenstein, with* $\operatorname{Socle}(R) = (tvx)$.
(3) *R is a Koszul algebra.*

Proposition 2.2 is proved in the appendix as Proposition A.4. A multiplication table for R is given in A.1 (via the isomorphism ψ defined in A.3.)

2.3. For each $i \le 0$ we let $d_i\colon R^2 \to R^2$ denote the map given with respect to the standard basis of R^2 by the matrix

$$d_i\ :\quad \begin{pmatrix} v & y \\ \alpha^{1-i}x & z \end{pmatrix}$$

Also, let $d_1\colon R^3 \to R^2$ denote the map represented with respect to the standard bases of R^3 and R^2 by the matrix

$$d_1\ :\quad \begin{pmatrix} v & y & 0 \\ x & z & tv \end{pmatrix}$$

Consider a minimal free resolution of $\operatorname{Coker} d_1$ with d_1 as the first differential:

$$\cdots \to R^{\beta_i} \xrightarrow{d_i} R^{\beta_{i-1}} \to \cdots \to R^{\beta_2} \xrightarrow{d_2} R^3 \xrightarrow{d_1} R^2$$

2.4. *Remark.* Looking at the multiplication table of R in A.1, one sees that the following elements are part of a minimal system of generators for $\operatorname{Ker} d_1$:

$$(0,0,t),\ (0,0,u),\ (0,0,v),\ (0,0,y),\ (0,0,z)$$

Thus $\beta_2 \ge 5$.

2.5. Theorem. *The sequence of homomorphisms*

$$\boldsymbol{C}:\quad \cdots \to R^{\beta_2} \xrightarrow{d_2} R^3 \xrightarrow{d_1} R^2 \xrightarrow{d_0} R^2 \xrightarrow{d_{-1}} R^2 \xrightarrow{d_{-2}} R^2 \to \cdots$$

is a minimal complete resolution such the following hold:

(1) *The sequence* $\{\operatorname{rank} C_i\}_{i\ge 0}$ *has exponential growth.*
(2) $\operatorname{rank} C_i = 2$ *for all* $i \le 0$.

Proof. We postpone the proof of (1) to the end of the section. The minimality of $\boldsymbol{C}$ is clear from the definition of the differentials d_i. Moreover, the defining equations of R guarantee $d_i d_{i+1} = 0$ for all $i \le 0$, hence $\boldsymbol{C}$ is a complex. Since the ring R is Gorenstein, the exactness of $\boldsymbol{C}$ also implies that the complex $\operatorname{Hom}_R(\boldsymbol{C}, R)$ is exact. Therefore it remains to show that $\boldsymbol{C}$ is exact.

Let (a, b) denote an element of R^2 written in the standard basis of R^2 as a free R-module. One may check that for each $i \le 0$ the k-vector space $\operatorname{Im} d_i$ has the following fourteen linearly independent elements:

$$\begin{aligned}
&d_i(1,0)=(v,\alpha^{1-i}x) && d_i(tv,0)=(0,\alpha^{1-i}tvx)\\
&d_i(t,0)=(tv,\alpha^{1-i}tx) && d_i(tx,0)=(tvx,0)\\
&d_i(u,0)=(uv,0) && d_i(0,1)=(y,z)\\
&d_i(v,0)=(0,\alpha^{1-i}vx) && d_i(0,t)=(vx-tv,tv-(\alpha+1)vx)\\
&d_i(x,0)=(vx,\alpha^{1-i}(tv+tx)) && d_i(0,u)=(0,\alpha uv+tx)\\
&d_i(y,0)=(vy,0) && d_i(0,v)=(vy,vz)\\
&d_i(z,0)=(vz,-\alpha^{2-i}vx) && d_i(0,y)=((\alpha-1)uv+tx,-vy).
\end{aligned}$$

We thus have $\operatorname{rank}_k \operatorname{Im} d_i \geq 14$ for all $i \leq 0$. Since $\operatorname{rank}_k R^2 = 28$, it follows that $\operatorname{rank}_k \operatorname{Ker} d_i \leq 14$ for all $i \leq 0$. We conclude that $H_i(\boldsymbol{C}) = 0$ for $i \leq -1$ and $\operatorname{rank}_k \operatorname{Ker} d_0 = 14$

For $i = 1$ the images above are also those of d_1, by replacing $d_i(1,0)$ with $d_1(1,0,0)$ and so on. Here (a,b,c) denotes an element of R^3 in its standard basis as a free R-module. Not more than thirteen of these images are linearly independent, since we have the relation

$$(vx, tv+tx) = (tv, tx) + (vx-tv, tv-(\alpha+1)vx) + (\alpha+1)(0,vx)\,.$$

One can check that we are left with precisely thirteen linearly independent elements, and a fourteenth element in $\operatorname{Im} d_1$ can be chosen to be $d_1(0,0,1) = (0,tv)$. Thus $\operatorname{rank}_k \operatorname{Im} d_1 \geq 14$ and it follows that $H_0(\boldsymbol{C}) = 0$. The definition of $\boldsymbol{C}_{\geq 1}$ gives that $H_i(\boldsymbol{C}) = 0$ for all $i \geq 1$, and therefore $\boldsymbol{C}$ is exact. □

Next we provide the necessary background for the proof of part (1) of the Theorem.

2.6. If $\pi\colon A \to B$ is a ring homomorphism, D is an A-module, E is a B-module (with the A-module structure induced by π) and $\phi\colon D \to E$ is a homomorphism of A-modules, then for each B-module L one has a natural homomorphism

$$\operatorname{Tor}^{\pi}(L,\phi)\colon\ \operatorname{Tor}^A(L,D) \to \operatorname{Tor}^B(L,E)$$

which may be computed as follows: Let $\boldsymbol{D}$ be a free A-resolution of D and $\boldsymbol{E}$ a free B-resolution of E. Let $\widetilde{\boldsymbol{\phi}}\colon B \otimes_A \boldsymbol{D} \to \boldsymbol{E}$ be a lifting of ϕ to a homomorphism of complexes of B-modules. The homomorphism $\operatorname{Tor}^{\pi}(L,\phi)$ is then induced in homology by the following homomorphism of complexes, which is unique up to homotopy:

$$L \otimes_A \boldsymbol{D} = L \otimes_B (B \otimes_A \boldsymbol{D}) \xrightarrow{L\otimes_B \widetilde{\boldsymbol{\phi}}} L \otimes_B \boldsymbol{E}$$

We say that a positively graded ring $A = \bigoplus_{i\geq 0} A_i$ is *standard graded* if A_0 is a field and A is generated over A_0 by A_1.

2.7. **Proposition.** *Let A be a standard graded algebra over a field ℓ and set $\mathfrak{n} = \bigoplus_{i\geq 1} A_i$. Let $\pi\colon A \to B$ be a surjective homomorphism of graded rings with $\operatorname{Ker}\pi \subseteq \mathfrak{n}^2$. Assume that D is a finitely generated graded A-module with a linear graded free resolution and that E is a finitely generated graded B-module.*

Suppose there exists a homomorphism of A-modules $\phi\colon D \to E$ such that the induced map $\overline{\phi}\colon D/\mathfrak{n}D \to E/\mathfrak{n}E$ is injective. Then the induced homomorphisms

$$\operatorname{Tor}^{\pi}_i(\ell,\phi)\colon\ \operatorname{Tor}^A_i(\ell,D) \to \operatorname{Tor}^B_i(\ell,E)$$

are injective for each i.

Proof. Consider a linear resolution $(\boldsymbol{D},\delta)$ of D, together with an augmentation map $\varepsilon\colon D_0 \to D$. We will construct inductively a minimal graded free resolution $\boldsymbol{E}$ of E, with an augmentation map $\eta\colon E_0 \to E$, and a map of complexes of A-modules $\boldsymbol{\varphi}\colon \boldsymbol{D} \to \boldsymbol{E}$ such that the induced map $B \otimes_A \boldsymbol{\varphi}\colon B \otimes_A \boldsymbol{D} \to \boldsymbol{E}$ is a split injection in each homological degree.

Let $e_1, \dots, e_a$ denote the standard basis of $D_0 = A^a$. The elements $\varepsilon(e_1), \dots, \varepsilon(e_a)$ form then a homogeneous minimal system of generators for D. Since the induced map $\overline{\phi}\colon D/\mathfrak{n}D \to E/\mathfrak{n}E$ is injective, the elements $\phi(\varepsilon(e_1)), \dots, \phi(\varepsilon(e_a))$ are part of a minimal system of generators for E. This shows that we can choose $E_0 = B^b$, with $b \geq a$, and the map $\eta\colon B^b \to E$ can be chosen so that $\eta(f_i) = \phi(\varepsilon(e_i))$ for each i with $1 \leq i \leq a$, where $f_1, \dots, f_b$ is the standard basis of B^b.

If we define the A-module homomorphism $\varphi_0\colon A^a \to B^b$ such that $\varphi_0(e_i) = f_i$, then the right-hand part of the diagram below is commutative. We set $D' = \operatorname{Ker}(\varepsilon)$ and $E' = \operatorname{Ker}(\eta)$, and we let $\phi'\colon D' \to E'$ denote the induced A-module homomorphism, which makes entire diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D' & \longrightarrow & A^a & \xrightarrow{\varepsilon} & D & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & E' & \longrightarrow & B^b & \xrightarrow{\eta} & E & \longrightarrow & 0 \end{array}$$

We want to prove that the induced map $\overline{\phi'}\colon D'/\mathfrak{n}D' \to E'/\mathfrak{n}E'$ is injective.

Let $g_1, \dots, g_{a'}$ be the standard basis of $D_1 = A^{a'}$. The elements $\delta_1(g_1), \dots, \delta_1(g_{a'})$ form a minimal system of generators for D'. Let $\alpha_i \in A$ be such that

$$\varphi_0\big(\alpha_1\delta_1(g_1) + \cdots + \alpha_{a'}\delta_1(g_{a'})\big) \in \mathfrak{n}E' \subseteq \mathfrak{n}^2B^b.$$

Since the matrix representing δ_1 has linear entries, we can think of $\delta_1(g_i)$ as column vectors with components in A_1 and hence of $\varphi_0(\delta_1(g_i))$ as column vectors with components in B_1. Thus, the degree one part of the above expression is equal to zero, so we get

$$\varphi_0\left(\sum_i \overline{\alpha}_i\delta_1(g_i)\right) = \sum_i \overline{\alpha}_i\varphi_0(\delta_1(g_i)) = 0$$

where $\overline{\alpha}_i$ denotes the degree zero component of α_i. Since the homomorphism $\pi\colon A \to B$ has $\operatorname{Ker}\pi \subseteq \mathfrak{n}^2$, we conclude that $\sum_i \overline{\alpha}_i\delta_1(g_i) \in \mathfrak{n}^2A^a$, hence, by degree considerations, $\sum_i \overline{\alpha}_i\delta_1(g_i) = 0$. Since the elements $\delta_1(g_i)$ in this sum are part of a homogeneous minimal system of generators for D', it follows that $\overline{\alpha}_i = 0$ for all i. Therefore $\alpha_i \in \mathfrak{n}$ for all i, and this shows $\overline{\phi'}$ is injective.

Using the construction above as the induction step, we obtain then a resolution $\boldsymbol{E}$ and a homomorphism of complexes $\boldsymbol{\varphi}\colon \boldsymbol{D} \to \boldsymbol{E}$. The homomorphism of complexes $\widetilde{\boldsymbol{\phi}} = \boldsymbol{\varphi} \otimes_A B\colon \boldsymbol{D} \otimes_A B \to \boldsymbol{E}$ is then a lifting of ϕ and is a split injection in each degree. This gives the desired conclusion. $\square$

Recall that P denotes the polynomial ring $k[T,U,V,X,Y,Z]$ and that the monomials U^2 and UY are among the generators of the ideal I defining R as P/I.

2.8. Lemma. *Consider the ring $A = P/(U^2,UY)$, the A-module $D = (U,Y)A$, and the R-module $E = \operatorname{Im} d_2$. Let $\pi\colon A \to R$ denote the canonical projection. The following then hold:*

(1) *The A-module D has a linear resolution and its Poincaré series is equal to $(2+t)(1-t-t^2)^{-1}$.*
(2) *There exists a homomorphism of A-modules $\phi\colon D \to E$ such that the induced map $\operatorname{Tor}_i^{\pi}(k,\phi)\colon \operatorname{Tor}_i^A(k,D) \to \operatorname{Tor}_i^R(k,E)$ is injective for each i.*

Proof. (1) Set $Q = k[U,Y]/(U^2, UY)$ and let $\boldsymbol{G}$ denote a minimal free resolution of the residue field $Q/(U,Y)Q$ over Q. Note that $A \cong Q \otimes_k k[T,V,X,Z]$, and a minimal free resolution of A/D over A is given by the complex $\boldsymbol{D} = \boldsymbol{G} \otimes_k k[T,V,X,Z]$.

Since Q is a Koszul algebra (see for example [5]), the resolution $\boldsymbol{G}$ is linear, and the Poincaré series of A/D over A is

$$\mathrm{P}^A_{A/D}(t) = \mathrm{P}^Q_{Q/(U,Y)Q}(t) = \frac{1}{\mathrm{H}_Q(-t)} = \frac{1+t}{1-t-t^2}\,.$$

The Poincaré series of D is then equal to $t^{-1}(\mathrm{P}^A_{A/D}(t) - 1)$.

(2) Set $p = (0,0,u)$ and $q = (0,0,y)$, considered as elements of R^3. It can be easily checked that $p, q \in \operatorname{Ker} d_1$, hence $Rp + Rq \subseteq E$. We define $\phi\colon D \to E$ as the following composition:

$$\phi\colon D \to Rp + Rq \hookrightarrow E,$$

where the leftmost map is the restriction of the map $\varphi\colon A \to R^3$ given by $\varphi(r) = \big(0,0,\pi(r)\big)$. Note that $ap + bq \in \mathfrak{m}E$ for some $a, b \in R$ implies $ap + bq \in \mathfrak{m}^2R^3$, and hence $a, b \in \mathfrak{m}$ by degree considerations. This shows that the induced map $\overline{\phi}\colon D/\mathfrak{n}D \to E/\mathfrak{n}E$ is injective, where $\mathfrak{n}$ denotes the maximal ideal of A. We can then apply Proposition 2.7. ◻

Proof of Theorem 2.5(1). We use the notation in the statement of the Lemma above. Part (1) of the Lemma shows that the sequence $\{\operatorname{rank}_k \operatorname{Tor}_i^A(k,D)\}_i$ has exponential growth. From part (2) we conclude that the sequence $\{\operatorname{rank}_k \operatorname{Tor}_i^R(k,E)\}_i$ has exponential growth, as well. Note that a minimal free resolution of the R-module E is given by the truncation $\boldsymbol{C}_{\geqslant 2}$, hence $\operatorname{rank}_k C_{i+2} = \operatorname{rank}_k \operatorname{Tor}_i^R(k,E)$ for all $i \geq 0$. ◻

3. Asymmetry in the vanishing of Ext

We will use the notation introduced in the second section. In particular, the ring R is the one defined in 2.1. Recall that R is zero-dimensional and Gorenstein.

3.1. If $\boldsymbol{T}$ is a complete resolution of the R-module X, and Y is an R-module, then for each i the *Tate (co)homology* groups are defined by

$$\widehat{\operatorname{Ext}}{}^i_R(X,Y) = \mathrm{H}_{-i}\operatorname{Hom}(\boldsymbol{T},Y) \quad\text{and}\quad \widehat{\operatorname{Tor}}{}^R_i(X,Y) = \mathrm{H}_i(\boldsymbol{T}\otimes_R Y)\,.$$

Since the ring R is zero-dimensional Gorenstein, the complete resolution $\boldsymbol{T}$ can be chosen to agree with a minimal free resolution of X in all nonnegative degrees, cf. [3, 3.1], for example. Thus for all $i > 0$ there are isomorphisms

$$\widehat{\operatorname{Ext}}{}^i_R(X,Y) \cong \operatorname{Ext}^i_R(X,Y) \quad\text{and}\quad \widehat{\operatorname{Tor}}{}^R_i(X,Y) \cong \operatorname{Tor}^R_i(X,Y)\,.$$

Also, for all i one has

$$\widehat{\operatorname{Ext}}{}^{-i-1}_R(X,Y) \cong \widehat{\operatorname{Tor}}{}^R_i(X^*,Y)\,.$$

Matlis duality yields for all i the following isomorphisms:

$$\operatorname{Tor}^R_i(Y,X^*)^* \cong \operatorname{Ext}^i_R(Y,X).$$

Recall that d_i denotes the differential of the complex $\boldsymbol{C}$ defined in Section 2.

3.2. Theorem. *Set* $M = \operatorname{Coker} d_0^*$ *and* $N = R/(t,u,v-x,y-x,z-x)$. *The following then hold:*

(1) $\widehat{\operatorname{Ext}}^i_R(M,N) = 0$ *for all* $i > 0$
(2) $\widehat{\operatorname{Ext}}^i_R(M,N) \neq 0$ *for all* $i < 0$.
(3) $\widehat{\operatorname{Tor}}^R_i(M,N) = 0$ *for all* $i > 0$.
(4) $\widehat{\operatorname{Tor}}^R_i(M,N) \neq 0$ *for all* $i < 0$.

In view of the isomorphisms in 3.1, we conclude the Gorenstein ring R does not have the property **(ee)**:

3.3. Corollary. *For R, M and N as above one has:*

$$\operatorname{Ext}^i_R(M,N) = 0 \text{ for all } i > 0\,;$$
$$\operatorname{Ext}^i_R(N,M) \neq 0 \text{ for all } i > 0.$$

The proof of the Theorem is given at the end of the section. We present below one of the ingredients of the proof.

3.4. Lemma. *Set* $E = \operatorname{Im} d_2$. *If L is an R-module of length two, then the following hold:*

(1) $\operatorname{Tor}^R_i(E,L) \neq 0$ *for all* $i > 0$.
(2) $\operatorname{Ext}^i_R(E,L) \neq 0$ *for all* $i > 0$.

Proof. (1) Since L has length two, there is an exact sequence

$$0 \to k \to L \to k \to 0.$$

Using the notation of Lemma 2.8, and the naturality of the maps defined in 2.6, the short exact sequence above induces long exact sequences both over R and over A, and they can be embedded in a commutative diagram as follows:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \operatorname{Tor}^A_i(k,D) & \longrightarrow & \operatorname{Tor}^A_i(L,D) & \longrightarrow & \operatorname{Tor}^A_i(k,D) & \xrightarrow{\Delta^A_i} & \operatorname{Tor}^A_{i-1}(k,D) & \longrightarrow \cdots \\
& \big\downarrow {\scriptstyle \operatorname{Tor}^\pi_i(k,\psi)} & & \big\downarrow {\scriptstyle \operatorname{Tor}^\pi_i(L,\psi)} & & \big\downarrow {\scriptstyle \operatorname{Tor}^\pi_i(k,\psi)} & & \big\downarrow {\scriptstyle \operatorname{Tor}^\pi_{i-1}(k,\psi)} & \\
\cdots \longrightarrow & \operatorname{Tor}^R_i(k,E) & \longrightarrow & \operatorname{Tor}^R_i(L,E) & \longrightarrow & \operatorname{Tor}^R_i(k,E) & \xrightarrow{\Delta^R_i} & \operatorname{Tor}^R_{i-1}(k,E) & \longrightarrow \cdots
\end{array}$$

Counting from the left we have: the first, third and fourth vertical maps are injective, cf. Lemma 2.8(2). If $\operatorname{Tor}^R_i(E,L) = 0$ for some $i > 0$, then the connecting homomorphism Δ^R_i is injective, so the commutativity of the rightmost square implies that Δ^A_i is injective. However, this is not possible, because Lemma 2.8(1) shows that the Betti numbers of D over A are strictly increasing.

(2) By Matlis duality, the R-module L^* has length two. We can then apply part (1) to conclude $\operatorname{Tor}^R_i(E,L^*) \neq 0$ for all $i > 0$ and then use the isomorphism $\operatorname{Ext}^i_R(E,L)^* \cong \operatorname{Tor}^R_i(E,L^*)$. □

Proof of Theorem 3.2. (1) A complete resolution of M is given by the complex $\operatorname{Hom}_R(\boldsymbol{C},R)$. We have

$$\widehat{\operatorname{Ext}}^i_R(M,N) = \operatorname{H}_{-i}(\operatorname{Hom}_R(\operatorname{Hom}_R(\boldsymbol{C},R),N)) \cong \operatorname{H}_{-i}(\boldsymbol{C} \otimes_R N)$$

In negative degrees $\boldsymbol{C} \otimes_R N$ is the complex

$$N^2 \xrightarrow{\left(\begin{smallmatrix} x & x \\ \alpha x & x \end{smallmatrix}\right)} N^2 \xrightarrow{\left(\begin{smallmatrix} x & x \\ \alpha^2 x & x \end{smallmatrix}\right)} N^2 \xrightarrow{\left(\begin{smallmatrix} x & x \\ \alpha^3 x & x \end{smallmatrix}\right)} N^2 \to \cdots$$

Since $N \cong k[x]/(x^2)$, this complex is acyclic, hence $\widehat{\operatorname{Ext}}^i_R(M,N) = 0$ for all $i > 0$.

(2) By the isomorphisms in 3.1 we need to show that $\operatorname{Tor}_i^R(M^*, N) \neq 0$ for all $i > 0$. Note that $M^* = \operatorname{Coker} d_2$, hence the module E in Lemma 3.4 is the first syzygy of M^*. Since N has length 2, the Lemma then shows $\operatorname{Tor}_i^R(M^*, N) \neq 0$ for all $i \geq 2$. To show that $\operatorname{Tor}_1^R(M^*, N) \neq 0$, consider the short exact sequence

$$0 \to k \to N \to k \to 0.$$

and the induced long exact sequence

$$\operatorname{Tor}_1^R(M^*, N) \to \operatorname{Tor}_1^R(M^*, k) \to M^* \otimes_R k \to M^* \otimes_R N \to M^* \otimes_R k \to 0$$

If $\operatorname{Tor}_1^R(M^*, N) = 0$, then $\beta_2 = \dim_k \operatorname{Tor}_1(M^*, N) \leq \dim_k(M^* \otimes_R k) = 3$, and this contradicts Remark 2.4.

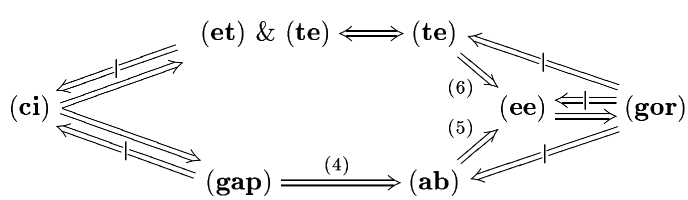
The proofs of (3) and (4) are similar. □

3.5. *Remark.* In [8] we gave an example of a codimension five Gorenstein ring R' which provides counterexamples to a conjecture of Auslander. More precisely, we proved that there exist finitely generated R'-modules M' and $\{N'_q\}_{q \geq 1}$, such that $\operatorname{Ext}^i_{R'}(M', N'_q) = 0$ if and only if $i \neq 0, q-1, q$. We also showed there exist finitely generated R'-modules L' such that $\operatorname{Tor}_i^{R'}(M', L') = 0$ for all $i \geq 1$, and $\operatorname{Ext}^i_{R'}(M', L') \neq 0$ for all $i \geq 1$.

For the ring R and the module M defined in this section it is easy to define R-modules N_q and L which give an analogous behavior. Indeed, the corresponding R-modules are $N_q = R/(t, u, v - \alpha^q x, v - y, v - z)$, and L the cokernel of the map $R^{10} \to R^2$ defined with respect to the standard bases of R^{10} and R^2 by the matrix

$$\begin{pmatrix} x & 0 & v & y & z & 0 & t & u & 0 & 0 \\ -z & x & -y & 0 & 0 & v & 0 & 0 & t & u \end{pmatrix}.$$

The ring R of this paper gives thus all the counterexamples to reverse implications in the right-hand side of the diagram in 4.6 in [8]. We reproduce this diagram below, with the added improvement given by this paper, namely that (**gor**) does not imply (**ee**). Here (**ci**) denotes the class of local complete intersection rings, (**gor**) the class of local Gorenstein rings, (**ee**) is the class of rings satisfying the property with the same name defined in the introduction, (**te**) is the class of rings for which $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \gg 0$ implies $\operatorname{Ext}^i_R(M,N) = 0$ for all $i \gg 0$, and (**ab**) is the class of local Gorenstein rings for which $\operatorname{Ext}^i_R(M,N) = 0$ for all $i \gg 0$ implies $\operatorname{Ext}^i_R(M,N) = 0$ for all $i > \dim R$.



We established thus that the homologically defined class (**ee**) is strictly contained between (**ci**) and (**gor**). We do not know whether the implications (4), (5), or (6) are reversible.

3.6. *Remark.* Subsequent to [8], Smalø [16], and independently Mori [12], gave a very simple non-commutative counterexample to the conjecture of Auslander, involving the ring $A = k\langle x, y\rangle/\langle x^2, y^2, xy - \alpha yx\rangle$, where $\alpha \in k$ has infinite multiplicative order. However, this ring fails to supply counterexamples to (**ee**). Indeed, all the indecomposable modules over A have been classified, see for example [14, Section 4]. They are A, syzygies and cosyzygies of k, modules of periodicity one, and non-periodic modules having bounded resolutions. The differentials in the resolutions of the latter type are described as follows: there exists a square matrix $M(t)$ with entries in $A[t]$, with t an indeterminate, such that the ith differential of the resolution is represented by $M(\alpha^i)$. An argument similar to that of [8, 3.13] may be used to see that the ring A satisfies (**ee**).

Appendix A

Let S denote the 14-dimensional k-vector space, with basis

$$\{1, a, b, c, d, e, f, l, m, n, p, q, r, s\}.$$

We set

$$\begin{aligned} h &= -(\alpha+1)m + p \\ i &= -(\alpha+1)n + l \\ j &= (\alpha-1)m + r \\ w &= r + \alpha m \end{aligned}$$

A.1. On the above basis we define a multiplication table as follows.

$\cdot$	1	a	b	c	d	e	f	l	m	n	p	q	r	s
1	1	a	b	c	d	e	f	l	m	n	p	q	r	s
a	a	h	0	l	r	$n-l$	i	0	0	s	0	0	0	0
b	b	0	0	m	0	0	w	0	0	0	0	s	0	0
c	c	l	m	0	n	p	q	0	0	0	0	0	s	0
d	d	r	0	n	$l+r$	0	$-\alpha n$	s	0	s	0	0	0	0
e	e	$n-l$	0	p	0	j	$-p$	0	0	0	s	0	0	0
f	f	i	w	q	$-\alpha n$	$-p$	0	0	s	0	0	0	$-\alpha s$	0
l	l	0	0	0	s	0	0	0	0	0	0	0	0	0
m	m	0	0	0	0	0	s	0	0	0	0	0	0	0
n	n	s	0	0	s	0	0	0	0	0	0	0	0	0
p	p	0	0	0	0	s	0	0	0	0	0	0	0	0
q	q	0	s	0	0	0	0	0	0	0	0	0	0	0
r	r	0	0	s	0	0	$-\alpha s$	0	0	0	0	0	0	0
s	s	0	0	0	0	0	0	0	0	0	0	0	0	0

We then define on S a multiplication by extending this operation by linearity. We want to prove that this operation defines a ring structure on S.

A.2. **Lemma.** *The vector space S, equipped with the multiplication described above, is a commutative ring.*

Proof. It is clear that the element 1 satisfies the identity axiom. Also, the operation is commutative, since the table in A.1 is symmetric. To prove that S is a ring, we only need to verify that the multiplication is associative.

We need to check that $g_1(g_2g_3) = (g_1g_2)g_3$ for all g_1, g_2, g_3 among the 14 basis elements. Note that we may assume $g_1 \neq 1$, $g_2 \neq 1$, $g_3 \neq 1$. If one of g_1, g_2, g_3 is equal to an element in the set $\{l, m, n, p, q, r, s\}$, then $g_1(g_2g_3) = (g_1g_2)g_3 = 0$. Thus we only need to verify the associativity relation when g_1, g_2, g_3 are among the elements a, b, c, d, e, f.

Consider the ordering $a \leq b \leq c \leq d \leq e \leq f$. In view of the commutativity relations, it suffices to show that $g_1(g_2g_3) = (g_1g_2)g_3 = (g_1g_3)g_2$ for all $g_1 \leq g_2 \leq g_3$. Since this is trivially satisfied when $g_1 = g_2 = g_3$, we only need to check 50 associations.

Note that we have the following relations:

$$ab = b^2 = bd = be = c^2 = de = f^2 = 0$$

Also, table A.1 yields the table below:

$\cdot$	a	b	c	d	e	f
a^2	0	0	0	0	s	$-(\alpha+1)s$
ac	0	0	0	s	0	0
ad	0	0	s	0	0	$-\alpha s$
ae	s	0	0	0	0	0
af	$-(\alpha+1)s$	0	0	$-\alpha s$	0	0
bc	0	0	0	0	0	s
bf	0	0	s	0	0	0
cd	s	0	0	s	0	0
ce	0	0	0	0	s	0
cf	0	s	0	0	0	0
d^2	0	0	s	s	0	$-\alpha s$
df	$-\alpha s$	0	0	$-\alpha s$	0	0
e^2	0	0	s	0	0	$-s$
ef	0	0	0	0	$-s$	0

Using the table and the relations above, we see that, among the 50 triples (g_1, g_2, g_3) that need to be checked for associativity, 42 of them give:

$$g_1(g_2g_3) = (g_1g_2)g_3 = (g_1g_3)g_2 = 0$$

The remaining 8 are as follows:

$$a(ae) = a^2e = (ae)a = s \qquad b(cf) = (bc)f = (bf)c = s$$
$$a(af) = a^2f = (af)a = -(\alpha+1)s \qquad c(d^2) = (cd)d = (cd)d = s$$
$$a(df) = (ad)f = (af)d = -\alpha s \qquad c(e^2) = (ce)e = (ce)e = s$$
$$d(df) = d^2f = (df)d = -\alpha s \qquad e(ef) = e^2f = (ef)e = -s$$

∎

We now make the connection with the ring R defined in 2.1.

A.3. Analyzing the defining equations of R given there, one can check that the elements

$$1,\ t,\ u,\ v,\ x,\ y,\ z,\ tv,\ uv,\ vx,\ vy,\ vz,\ tx,\ vtx$$

generate R as a vector space over k, hence the elements t, u, v, x, y, z generate R as a k-algebra. Let $\psi\colon R \to S$ denote the k-algebra homomorphism defined by

$$\psi(t) = a,\ \psi(u) = b,\ \psi(v) = c,\ \psi(x) = d,\ \psi(y) = e,\ \psi(z) = f$$

Note that we also have:

$$\psi(tv) = l,\ \psi(uv) = m,\ \psi(vx) = n,\ \psi(vy) = p,\ \psi(vz) = q,\ \psi(tx) = r,\ \psi(vtx) = s\,.$$

A.4. **Proposition.** *The following hold:*

(1) *The ring S is a Gorenstein local ring with maximal ideal $\mathfrak{n} = (a, b, c, d, e, f)$, Hilbert series $1 + 6t + 6t^2 + t^3$, and* $\operatorname{Socle} S = (s)$.
(2) *The homomorphism of k-algebras $\psi\colon R \to S$ defined above is an isomorphism.*
(3) *The elements listed in* A.3 *form a basis of R over k.*
(4) *The graded ring R is a Koszul algebra.*

Proof. (1) Note that $\mathfrak{n}^2 = (l, m, n, p, q, r)$, $\mathfrak{n}^3 = (s)$ and $\mathfrak{n}^4 = 0$. Since $S/\mathfrak{n}S = k$ is a field, we know that $\mathfrak{n}$ is a maximal ideal, and since it is nilpotent, it is unique. We have $(s) \subseteq \operatorname{Socle}(S)$ and the multiplication table in A.1 shows that equality holds. Since its socle is 1-dimensional, the ring S is Gorenstein.

(2) The map ψ is clearly surjective. Recall that the elements listed in A.3 generate R as a vector space over k. Noting that the images of these elements through ψ are linearly independent, we conclude that ψ is injective.

(3) This is now clear.

(4) In 2.1 we have written the generators of I so that the first monomial occurring in each generator is its initial term with respect to reverse lexicographic order associated to the variable ordering $Z > U > Y > X > T > V$. Let J denote the ideal generated by these initial terms:

$$J = (Z^2, UZ, U^2, YZ, UY, Y^2, XZ, UX, XY, X^2, TZ, TU, TY, T^2, V^2)\,.$$

It is easy to check that the Hilbert series of P/J is equal to $1 + 6t + 6t^2 + t^3$, and hence it is equal to the Hilbert series of R. It follows that the initial ideal of I equals J, and this shows that the generators of I listed above are a Gröbner basis for I. By [5] this shows that R is a Koszul algebra. □

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