FREE RESOLUTIONS OVER SHORT GORENSTEIN LOCAL RINGS

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ABSTRACT. Let R be a local ring with maximal ideal $\mathfrak{m}$ admitting a non-zero element $a \in \mathfrak{m}$ for which the ideal $(0 : a)$ is isomorphic to R/aR. We study minimal free resolutions of finitely generated R-modules M, with particular attention to the case when $\mathfrak{m}^4 = 0$. Let e denote the minimal number of generators of $\mathfrak{m}$. If R is Gorenstein with $\mathfrak{m}^4 = 0$ and $e \geq 3$, we show that $\mathrm{P}_M^R(t)$ is rational with denominator $\mathrm{H}_R(-t) = 1 - et + et^2 - t^3$, for each finitely generated R-module M. In particular, this conclusion applies to generic Gorenstein algebras of socle degree 3.

INTRODUCTION

Let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k, and let M be a finite (meaning finitely generated) R-module. We study the minimal free resolution of M over R by means of the *Poincaré series* of M, which is the formal power series

$$\mathrm{P}_M^R(t) = \sum_{n=0}^{\infty} \mathrm{rank}_k \mathrm{Tor}_n^R(M, k) t^n \,.$$

The *Hilbert series* of M is the formal power series

$$\mathrm{H}_M(t) = \sum_{n=0}^{\infty} \mathrm{rank}_k \left(\frac{\mathfrak{m}^n M}{\mathfrak{m}^{n+1} M} \right) t^n.$$

In this paper we identify a class of local rings R with $\mathfrak{m}^4 = 0$ with the property that $\mathrm{H}_R(-t) \mathrm{P}_M^R(t)$ is a polynomial in $\mathbb{Z}[t]$ for every finite R-module M; in particular, $\mathrm{P}_M^R(t)$ is a rational function in t.

Research on the rationality of Poincaré series has uncovered the existence of both *bad* and *good* behavior. Anick [1] constructed the first example of a local ring R for which $\mathrm{P}_k^R(t)$ is irrational. Jacobsson [14] produced a local ring R and finite R-module M such that $\mathrm{P}_k^R(t)$ is rational, but $\mathrm{P}_M^R(t)$ is irrational. More recently, Roos [19] constructed a local ring R and a family of finite R-modules $\{M_i\}_{i \geq 0}$ such that the series $\mathrm{P}_{M_i}^R(t)$ is rational for all $i \geq 0$, but do not admit a common denominator. Remarkably, in all these examples the rings are short, in the sense that they satisfy $\mathfrak{m}^3 = 0$. On the other hand, there is a growing gallery of classes of rings R which are *good*, in the sense of [19], that is, there exists $d_R(t) \in \mathbb{Z}[t]$ such

that $d_R(t) \, \mathrm{P}_M^R(t) \in \mathbb{Z}[t]$ for all finite R-modules M. Among such rings are those with $\mu(\mathfrak{m}) - \operatorname{depth} R \leq 3$ and the Gorenstein rings R with $\mu(\mathfrak{m}) - \operatorname{depth} R = 4$, cf. [4] and [15], where $\mu(\mathfrak{m})$ denotes the minimal number of generators of $\mathfrak{m}$.

All Gorenstein rings $(R, \mathfrak{m}, k)$ with $\mathfrak{m}^3 = 0$ are known to be good, due to Sjödin [22]; see also Avramov, Iyengar, Şega [5]. However, Gorenstein local rings with $\mathfrak{m}^4 = 0$ can exhibit bad behavior, as there exist examples with $\mu(\mathfrak{m}) \geq 12$ for which $\mathrm{P}_k^R(t)$ is irrational (cf. Bøgvad [7]).

We state below our main result, in the primary case of interest.

Theorem. *Let $(R, \mathfrak{m}, k)$ be Gorenstein with $\mathfrak{m}^4 = 0$ and $\mu(\mathfrak{m}) = e \geq 3$.*

If there exists a non-zero element $a \in \mathfrak{m}$ such that the ideal $(0 :_R a)$ is principal, then the following hold:

(1) $\mathrm{H}_R(t) = 1 + et + et^2 + t^3$;
(2) $\mathrm{H}_R(-t) \, \mathrm{P}_k^R(t) = 1$;
(3) $\mathrm{H}_R(-t) \, \mathrm{P}_M^R(t)$ *is in $\mathbb{Z}[t]$, for every finite R-module M.*

The hypothesis of our theorem is verified by 'generic' Gorenstein rings with $\mathfrak{m}^4 = 0$ and $\mu(\mathfrak{m}) \geq 3$. Indeed, the set of Gorenstein standard graded k-algebras R with $\mathrm{H}_R(t) = 1 + et + et^2 + t^3$ can be parametrized by a projective space by means of the inverse systems correspondence of Macaulay, which associates to each such ring a form of degree 3. With this parametrization, Conca, Rossi and Valla [10] show that a generic Gorenstein graded k-algebra R as above contains a non-zero element $a \in \mathfrak{m}$ for which the ideal $(0 :_R a)$ is principal. They also establish (2), which is equivalent to saying that R is a Koszul algebra.

While the most significant statement of our results is given for Gorenstein rings, in the body of the paper we work with larger classes of rings. We say that an element $a \in R$ is an *exact zero divisor* if $R \neq (0 :_R a) \cong R/aR \neq 0$. Note that an element a satisfying the hypothesis of the Theorem is an exact zero divisor.

The paper is organized as follows. In sections 1 and 2 we show that for an exact zero divisor a some homological properties of R and R/aR are closely linked, and so are the Hilbert series of these rings.

In section 3, we establish our main result, Theorem 3.3. The rings considered are proved to be Koszul homomorphic images of a codimension two complete intersection ring under a Golod homomorphism. Results of Herzog and Iyengar [13] then yield that every finite module over R has a Koszul syzygy module. (See 1.10 for the definition of a Koszul module.) In the last section we study Gorenstein rings and deduce the theorem above.

1. Exact zero divisors and Poincaré series

In this section we introduce the notion of exact zero divisor and study some of its properties. The main results are Theorems 1.7 and 1.9. Wide-ranging generalizations of the results of this section are obtained in [5].

We start with a remark concerning ideals with cyclic annihilators. A version of the equivalence of (1) and (3), which does not require commutativity, is noted by Nicholson [18], who calls an element a satisfying these conditions *morphic*.

Remark 1.1. For a commutative ring R and for $a \in R$ the following are equivalent:

(1) $(0 : a) \cong R/aR$;

(2) there exists $b \in R$ such that the sequence of homomorphisms

$$\mathbf{F}_{a,b}(R) = \cdots \longrightarrow R \xrightarrow{b} R \xrightarrow{a} R \xrightarrow{b} R \xrightarrow{a} R \to 0$$

is a free resolution of R/aR over R;

(3) there exists $b \in R$ such that $(0 :_R a) = bR$ and $(0 :_R b) = aR$

Indeed, (1) implies $(0 :_R a) = bR$ for some $b \in R$, and $bR \cong R/aR$ yields $(0 :_R b) = aR$, so (3) holds. The implications $(3) \implies (2) \implies (1)$ are evident.

Remark 1.2. When $(R, \mathfrak{m}, k)$ is a local Noetherian ring, the equivalence of the conditions above for $a \in \mathfrak{m}$ is also proved by Soto. In addition, he proves that they are also equivalent to the freeness of the R/aR-module $(0 :_R a)$; see [23, Prop. 1].

Definition. Let R be a commutative ring. An element $a \in R$ is said to be an *exact zero divisor* if it satisfies

(1.2.1) $$R \neq (0 :_R a) \cong R/aR \neq 0.$$

We call any element b provided by Remark 1.1 a *complementary divisor* for a, and say that a, b form an *exact pair of zero divisors*. Note that if R is local then b is unique up to a unit.

1.3. *Exact zero divisors under flat homomorphisms.* Let $\varphi \colon R \to S$ be a flat ring homomorphism. It follows from the discussion above that the images under φ of an exact pair of zero divisors in R form an exact pair of zero divisors in S provided they are non-zero.

Example 1.4. Let $(P, \mathfrak{p}, k)$ be a local ring and u, v, w be elements in $\mathfrak{p}$. If uv, w is a regular sequence, then the images of u, v in $Q = P/(uv, w)$ form an exact pair of zero divisors, as seen by direct computation.

Lemma 1.5. *Let R be a commutative ring and M, N be R-modules. If $a \in R$ satisfies $aM = 0$ and $N/aN \cong (0 :_N a)$, then there exists an isomorphism of R-modules:*

$$\mathrm{Hom}_{R/aR}(M, N/aN) \cong \mathrm{Hom}_R(M, N).$$

In particular, if R is local with maximal ideal $\mathfrak{m}$ and a is an exact zero divisor, then

$$(0 :_{\overline{R}} \overline{\mathfrak{m}}) \cong (0 :_R \mathfrak{m})$$

where $\overline{R} = R/aR$ and $\overline{\mathfrak{m}} = \mathfrak{m}/aR$.

Proof. The following R-module isomorphisms are R/aR-linear:

$$N/aN \cong (0 :_N a) \cong \mathrm{Hom}_R(R/aR, N).$$

We have thus:

$$\begin{aligned}
\mathrm{Hom}_{R/aR}(M, N/aN) &\cong \mathrm{Hom}_{R/aR}(M, \mathrm{Hom}_R(R/aR, N)) \\
&\cong \mathrm{Hom}_R(M \otimes_{R/aR} R/aR, N) \\
&\cong \mathrm{Hom}_R(M, N)
\end{aligned}$$

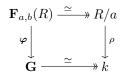where the second row is given by Hom-Tensor adjointness.

The last conclusion is obtained by taking $M = R/\mathfrak{m}$ and $N = R$. $\square$

Remark 1.6. Lemma 1.5 implies that if the ring R is artinian and a is an exact zero divisor, then R is Gorenstein if and only if $\overline{R} = R/aR$ is Gorenstein. This statement holds more generally without the assumption that R is artinian, see [23, Thm 1].
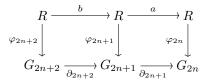
Theorem 1.7. *Let $(R, \mathfrak{m}, k)$ be a local ring and a, b be elements in $\mathfrak{m} \smallsetminus \mathfrak{m}^2$ forming an exact pair of zero divisors. For every finite R/aR-module M one has*

$$(1.7.1) \qquad \mathrm{P}_M^R(t) = \mathrm{P}_M^{R/aR}(t) \cdot \frac{1}{1-t}$$

Proof. Let $\mathbf{F}_{a,b}(R)$ be as in Remark 1.1 and $\mathbf{G}$ be a minimal free resolution of k over R. The canonical map $\rho \colon R/aR \to k$ extends to a morphism of complexes $\boldsymbol{\varphi} \colon \mathbf{F}_{a,b}(R) \to \mathbf{G}$, making the following diagram commute:

$$\begin{array}{ccc} \mathbf{F}_{a,b}(R) & \xrightarrow{\;\simeq\;} & R/a \\ {\scriptstyle\boldsymbol{\varphi}}\big\downarrow & & \big\downarrow{\scriptstyle\rho} \\ \mathbf{G} & \xrightarrow{\;\simeq\;} & k \end{array}$$

We show that $\varphi_n \otimes_R k$ is injective by induction on n where φ_n is the nth component of $\boldsymbol{\varphi}$. Indeed, if $\varphi_{2n+1}(r) \in \mathfrak{m}G_{2n+1}$ for some $r \in R$ then by minimality of $\mathbf{G}$, $\partial_{2n+1}(\varphi_{2n+1}(r)) = a \cdot \varphi_{2n}(r) \in \mathfrak{m}^2 G_{2n}$.

$$\begin{array}{ccccc} R & \xrightarrow{\;\;b\;\;} & R & \xrightarrow{\;\;a\;\;} & R \\ {\scriptstyle\varphi_{2n+2}}\big\downarrow & & {\scriptstyle\varphi_{2n+1}}\big\downarrow & & {\scriptstyle\varphi_{2n}}\big\downarrow \\ G_{2n+2} & \xrightarrow[\partial_{2n+2}]{} & G_{2n+1} & \xrightarrow[\partial_{2n+1}]{} & G_{2n} \end{array}$$

Since a is part of a minimal generating set for $\mathfrak{m}$ we conclude that $\varphi_{2n}(r) \in \mathfrak{m}G_{2n}$. By the induction hypothesis the map $\varphi_{2n} \otimes_R k$ is injective, hence r is in $\mathfrak{m}$. Thus, $\varphi_{2n+1} \otimes_R k$ is injective. A similar argument asserts that $\varphi_{2n+2} \otimes_R k$ is injective, concluding the induction step. The base of induction is clear. In this way we have that the map

$$\rho_* \colon \mathrm{Tor}_*^R(R/aR, k) \longrightarrow \mathrm{Tor}_*^R(k, k) \quad \text{is injective.}$$

Thus, as established by Levin in [17, 1.1, (3) $\Longrightarrow$ (2)],

$$\mathrm{P}_M^R(t) = \mathrm{P}_M^{R/aR}(t) \cdot \mathrm{P}_{R/aR}^R(t).$$

To conclude the proof note that $\mathrm{P}_{R/aR}^R(t) = (1-t)^{-1}$, since $\mathbf{F}_{a,b}(R)$ is a minimal free resolution of R/aR over R. $\qquad\square$

We establish next a graded version of Theorem 1.7. To do this we first fix some standard terminology for graded rings and modules.

1.8. *Graded Poincaré series.* Let k be a field and let $R = \oplus_{n \geq 0} R_n$ be a *standard graded k-algebra* (in the sense of [13, 1.8]). Recall that each graded module M over R has a graded free resolution and any such resolution induces a natural grading on

$$\mathrm{Tor}_i^R(M, k) = \bigoplus_j \mathrm{Tor}_i^R(M, k)_j.$$

One defines the *graded Betti numbers of M over R* to be

$$\beta_{i,j}^R(M) = \operatorname{rank}_k \ \operatorname{Tor}_i^R(M,k)_j$$

and the *graded Poincaré series of M over R* is defined to be

$$\mathrm{P}_M^R(s,t) = \sum_{i,j} \beta_{i,j}^R(M) \ s^i t^j$$

Thus $\mathrm{P}_M^R(s,t)$ is a formal power series in t with coefficients in the Laurent polynomial ring in s:

$$\mathrm{P}_M^R(s,t) \in \mathbb{Z}\left[s^{\pm 1}\right] \llbracket t \rrbracket$$

As in [20, §2], we say that a finite graded module M over a standard graded k-algebra R has a *linear resolution* if $\operatorname{Tor}_i^R(M,k)_j = 0$ for $j \neq i$; equivalently, if the graded Poincaré series $\mathrm{P}_M^R(s,t)$ can be written as a formal power series in the product st.

Theorem 1.9. *Let R be a standard graded k-algebra and a,b be elements in R_1 forming an exact pair of zero divisors on R. For every finite positively graded R/aR-module M, one then has*

$$(1.9.1) \qquad \mathrm{P}_M^R(s,t) = \mathrm{P}_M^{R/aR}(s,t) \cdot \frac{1}{1-st}$$

Proof. The proof of this theorem is a graded version of the previous proof. A bookkeeping exercise shows that the proof in [17, 1.1, (3)⇒(2)] translates smoothly to the graded case. Namely, that argument uses spectral sequences that preserve grading and lead to the following conclusion:

$$\mathrm{P}_M^R(s,t) = \mathrm{P}_M^{R/aR}(s,t) \cdot \mathrm{P}_{R/aR}^R(s,t) = \mathrm{P}_M^{R/aR}(s,t) \cdot \frac{1}{1-st} \ . \qquad \square$$

We discuss next the Koszul property for rings and modules. We refer the reader to [13] and [20] for detailed studies of the Koszul property.

Throughout we adopt the following notation for the associated graded objects over a local ring $(R,\mathfrak{m},k)$: let R^{g} denote the associated graded ring and let M^{g} denote the associated graded module of an R-module M; that is,

$$R^{\mathsf{g}} = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} \quad \text{and} \quad M^{\mathsf{g}} = \bigoplus_{n \geq 0} \mathfrak{m}^n M/\mathfrak{m}^{n+1}M.$$

1.10. *Koszul modules.* As in [13], we say that a finite module M over a local ring $(R,\mathfrak{m},k)$ is *Koszul* if for a minimal free resolution $\mathbf{F}$ of M the complex

$$\operatorname{lin}^R(F) = \ \cdots \longrightarrow F_{n+1}{}^{\mathsf{g}}(-n-1) \longrightarrow F_n{}^{\mathsf{g}}(-n) \longrightarrow \cdots \longrightarrow F_0{}^{\mathsf{g}} \longrightarrow M^{\mathsf{g}} \longrightarrow 0$$

with differentials induced from $\mathbf{F}$ is acyclic. A local ring R is said to be *Koszul* if its residue field k is Koszul as a module over R.

From [20, 2.3] and [13, 1.5] the following statements are equivalent for a finite module M over a local ring R:

(1) M is a Koszul R-module
(2) M^{g} has a linear resolution over R^{g}
(3) $\operatorname{Tor}_i^{R^{\mathsf{g}}}(M^{\mathsf{g}},k)_j = 0$ for $j \neq i$
(4) $\mathrm{P}_{M^{\mathsf{g}}}^{R^{\mathsf{g}}}(s,t)$ can be written as a formal power series in the product st.

Further, as proved in [13, Prop. 1.8], the Poincaré series of a Koszul module M over a local ring R satisfies

(1.10.1) $\mathrm{P}_M^R(t) = \mathrm{P}_{M^{\mathsf{g}}}^{R^{\mathsf{g}}}(t) = \mathrm{H}_M(-t)\,\mathrm{H}_R(-t)^{-1}.$

A standard graded k-algebra R is said to be a *Koszul algebra* if k admits a linear free resolution as a graded module over R. It follows that a local ring R is Koszul if and only if R^{g} is Koszul (as a standard graded k-algebra). Consequently, a local ring is Koszul if and only if $\mathrm{P}_{k^{\mathsf{g}}}^{R^{\mathsf{g}}}(1,t) = \mathrm{H}_R(-t)^{-1}$, as shown in [11, Thm 1].

1.11. *Graded Koszul modules.* A graded module M over a standard graded k-algebra R is said to be Koszul if for a minimal graded free resolution $\mathbf{F}$ of M, letting $\mathfrak{m}$ denote the irrelevant maximal ideal of R, the complex of $R_{\mathfrak{m}}$-modules $\mathrm{lin}^R(F_{\mathfrak{m}})$ is acyclic. If a graded R-module M has a linear resolution, then it is Koszul, but the converse fails in general; see [13, 1.9].

Corollary 1.12. *Let R be a standard graded k-algebra and a, b be elements in R_1 forming an exact pair of zero divisors on R. A finite positively graded R/aR-module M has a linear resolution over R if and only if it has a linear resolution over R/aR. In particular, the ring R is Koszul if and only if the ring R/aR is Koszul.*

Proof. Set $\overline{R} = R/aR$. As a consequence of Theorem 1.9, $\mathrm{P}_M^R(s,t)$ is a power series in the product st precisely when $\mathrm{P}_M^{\overline{R}}(s,t)$ is a power series in the product st. In this way, it becomes clear that M has a linear resolution over R if and only if it has one over $\overline{R}$. It follows immediately from the case $M = k$, that the ring R is Koszul if and only if the ring R/aR is Koszul. $\square$

2. Short local rings with balanced Hilbert series

In this section we study properties of exact pairs of zero divisors over local rings R satisfying $\mathfrak{m}^4 = 0$ and $H_R(-1) = 0$. Theorem 2.5 establishes that the initial forms a^*, b^* of an exact pair of zero divisors a, b in R remain an exact pair of zero divisors in R^{g}, yielding a numerical relation between the Hilbert series of R and R/aR.

We consider both local and graded rings, hence we proceed to clarify terminology. If R is a standard graded k-algebra and $M = \oplus_{n \geq s} M_n$ is a finite R-module, then the Hilbert series of M is the formal power series

$$\mathrm{H}_M(t) = \sum_{n \geq s} \mathrm{rank}_k(M_n) t^n \,.$$

If R is artinian, then R is in particular a local ring and this notion of Hilbert series coincides with the one defined for a local ring, whenever the module is generated in degree 0. In this setting, we will not distinguish between the local notion and the graded notion.

We say that a module M has *balanced Hilbert series* if $H_M(-1) = 0$.

Lemma 2.1. *Let R be a standard graded k-algebra and $a \in R_1$.*

If M is a finite graded R-module such that there exists an isomorphism of graded R-modules $M/aM(-1) \cong (0 :_M a)$, then

(2.1.1) $\mathrm{H}_M(t) = (1 + t)\,\mathrm{H}_{M/aM}(t).$

In particular, if $a, b \in R_1$ form an exact pair of zero divisors then (2.1.1) holds with $M = R$, hence R has balanced Hilbert series.

Proof. The additivity property of the Hilbert series in the exact sequence

$$0 \to (0 :_M a)(-1) \to M(-1) \xrightarrow{a} M \to M/aM \to 0$$

gives an equality

$$t \, \mathrm{H}_{(0:_M a)}(t) - t \, \mathrm{H}_M(t) + \mathrm{H}_M(t) - \mathrm{H}_{M/aM}(t) = 0 \,.$$

From the isomorphism $M/aM(-1) \cong (0 :_M a)$ of graded R-modules we then have an equality

$$(1 - t) \, \mathrm{H}_M(t) = (1 - t^2) \, \mathrm{H}_{M/aM}(t)$$

which establishes (2.1.1)

If $a, b \in R_1$ form an exact pair of zero divisors, the complex $\mathbf{F}_{a,b}(R)$ is exact and gives an isomorphism $R/aR(-1) \cong (0 :_R a)$ of graded R-modules yielding the desired conclusion. $\square$

Let $(R, \mathfrak{m}, k)$ be a local ring. When M is a finite R-module we let $\mu(M)$ denote the minimal number of generators of M over R; thus, one has $\mu(M) = \mathrm{rank}_k M/\mathfrak{m}M$. Also, $\lambda(M)$ denotes the length of M. When $\mathfrak{m}M = 0$, one has $\mu(M) = \lambda(M)$.

We collect below several basic facts used throughout.

Remark 2.2. Let $(R, \mathfrak{m}, k)$ be a local ring and $a \in \mathfrak{m}$. The following then hold:

(1) If $a \notin \mathfrak{m}^2$, then $aR \cap \mathfrak{m}^2 = a\mathfrak{m}$.
(2) If $a \neq 0$, then $\lambda(aR) = \lambda(a\mathfrak{m}^2) + \mu(a\mathfrak{m}) + 1$.
(3) $\lambda(0 :_R a) = \lambda(R) - \lambda(aR)$.
(4) If $\mathfrak{m}^4 = 0$, then $\lambda(R) = 1 + \mu(\mathfrak{m}) + \mu(\mathfrak{m}^2) + \mu(\mathfrak{m}^3)$.

Indeed, to prove (1), let $x \in R$ such that $ax \in \mathfrak{m}^2$. Since a is not contained in $\mathfrak{m}^2$, we conclude that x is not an unit, hence $ax \in a\mathfrak{m}$. Formula (2) is given by a length count in the exact sequences:

$$0 \to a\mathfrak{m}^2 \to a\mathfrak{m} \to a\mathfrak{m}/a\mathfrak{m}^2 \to 0$$

$$0 \to a\mathfrak{m} \to aR \to aR/a\mathfrak{m} \to 0 \,.$$

Formula (3) is given by a length count in the exact sequence

$$0 \to (0 :_R a) \to R \to aR \to 0 \,,$$

and (4) by a length count on the quotients of the $\mathfrak{m}$-adic filtration:

$$0 = \mathfrak{m}^4 \subset \mathfrak{m}^3 \subset \mathfrak{m}^2 \subset \mathfrak{m} \subset R \,.$$

Proposition 2.3. *Let $(R, \mathfrak{m}, k)$ be a local ring with $\mathfrak{m}^4 = 0$ and balanced Hilbert series. For a, b in $\mathfrak{m} \smallsetminus \mathfrak{m}^2$, the following statements are equivalent:*

(1) *a, b is an exact pair of zero-divisors*
(2) *$(0: a) = bR$*
(3) *$ab = 0$, $a\mathfrak{m}^2 = \mathfrak{m}^3 = b\mathfrak{m}^2$ and $\mu(a\mathfrak{m}) = \mu(\mathfrak{m}) - 1 = \mu(b\mathfrak{m})$.*

Proof. It is clear that (1) implies (2). For the rest it suffices to show that (2) and (3) are equivalent; indeed, if this is known, we see that (2) implies $(0: b) = aR$, since (3) is symmetric with respect to a and b.

We may assume $ab = 0$ as this holds under any of the conditions (1) to (3).

Set $e = \mu(\mathfrak{m})$ and $s = \mu(\mathfrak{m}^3)$. Since $\mathfrak{m}^4 = 0$, one gets $\mu(a\mathfrak{m}^2) = \lambda(a\mathfrak{m}^2)$ and $\mu(b\mathfrak{m}^2) = \lambda(b\mathfrak{m}^2)$. Remark 2.2(2) gives

$$(2.3.1) \qquad \lambda(aR) = \mu(a\mathfrak{m}^2) + \mu(a\mathfrak{m}) + 1 \,.$$

Since $b \notin \mathfrak{m}^2$, we may choose $b, x_2, x_3, \ldots, x_e$ to be a minimal system of generators for $\mathfrak{m}$. As $ab = 0$, we see that $a\mathfrak{m}$ is generated by $ax_2, \ldots, ax_e$, hence

$$(2.3.2) \qquad\qquad\qquad \mu(a\mathfrak{m}) \leq e - 1 \,.$$

From the inclusion $a\mathfrak{m}^2 \subseteq \mathfrak{m}^3$ we obtain the obvious inequality

$$(2.3.3) \qquad\qquad\qquad \mu(a\mathfrak{m}^2) \leq s$$

with equality if and only if $a\mathfrak{m}^2 = \mathfrak{m}^3$. Using (2.3.2) and (2.3.3) in (2.3.1) we thus have

$$(2.3.4) \qquad\qquad\qquad \lambda(aR) \leq e + s$$

with equality if and only if $\mu(a\mathfrak{m}) = e - 1$ and $a\mathfrak{m}^2 = \mathfrak{m}^3$. Similarly, we obtain

$$(2.3.5) \qquad\qquad\qquad \lambda(bR) \leq e + s$$

with equality if and only if $\mu(b\mathfrak{m}) = e - 1$ and $b\mathfrak{m}^2 = \mathfrak{m}^3$. The hypothesis $\mathrm{H}_R(-1) = 0$ implies $\lambda(R) = 2(e + s)$ and Remark 2.2(3) gives

$$\lambda(0 : a) = 2(e + s) - \lambda(aR) \,.$$

Thus the inequality (2.3.4) yields

$$(2.3.6) \qquad\qquad\qquad \lambda(0 : a) \geq e + s$$

with equality if and only if $\mu(a\mathfrak{m}) = e - 1$ and $a\mathfrak{m}^2 = \mathfrak{m}^3$.

Since $ab = 0$, we have $bR \subseteq (0 : a)$ and the inequalities above yield

$$\lambda(bR) \leq e + s \leq \lambda(0 : a).$$

We have thus $bR = (0 : a)$ if and only if equalities hold in (2.3.5) and (2.3.6), that is, if and only if condition (3) is satisfied. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2.4. *Let $(R, \mathfrak{m}, k)$ be a local ring satisfying $\mathfrak{m}^4 = 0$ and $\mu(\mathfrak{m}) = e \geq 3$.*

(1) *If R has balanced Hilbert series and an exact zero divisor lies in $\mathfrak{m} \smallsetminus \mathfrak{m}^2$ then so do its complementary divisors.*

(2) *If $\mu(\mathfrak{m}^3) + 2 \leq e$, then a non-zero exact zero divisor is not contained in $\mathfrak{m}^2$.*

Proof. Let $f = \mu(\mathfrak{m}^2)$ and $g = \mu(\mathfrak{m}^3)$.

(1) As $\mathfrak{m}^4 = 0$, we have $\mathrm{H}_R(t) = 1 + et + ft^2 + gt^3$. If $\mathrm{H}_R(-1) = 0$ then $g = f - e + 1$. Now let $a \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ be an exact zero divisor and b its complementary divisor.

Assume that b is in $\mathfrak{m}^2$. One then has

$$aR = (0 :_R b) \supseteq (0 :_R \mathfrak{m}^2) \supseteq \mathfrak{m}^2.$$

Hence $\mathfrak{m}^2 = a\mathfrak{m}$ by Remark 2.2 (1) and

$$f = \mu(\mathfrak{m}^2) = \mu(a\mathfrak{m}) \leq \mu(\mathfrak{m}) = e.$$

We have thus $g = f - e + 1 \leq 1$. Since $e \geq 3$, a contradiction is derived from (2).

(2) Assume b is a non-zero exact zero divisor in $\mathfrak{m}^2$ and let a be a complementary zero divisor. Then $b\mathfrak{m} \subseteq \mathfrak{m}^3$ hence

$$\lambda(bR) = \lambda(bR/b\mathfrak{m}) + \lambda(b\mathfrak{m}) \leq \mu(bR) + \lambda(\mathfrak{m}^3) = 1 + g.$$

On the other hand, $a\mathfrak{m} \subseteq \mathfrak{m}^2$ as $a \in \mathfrak{m}$, so

$$\lambda(aR) = \lambda(aR/a\mathfrak{m}) + \lambda(a\mathfrak{m}) \leq \mu(aR) + \lambda(\mathfrak{m}^2) = 1 + \mu(\mathfrak{m}^2) + \mu(\mathfrak{m}^3) = 1 + f + g.$$

From Remark 2.2 (4) and (3), we have

$$1 + e + f + g = \lambda(R) = \lambda(aR) + \lambda(bR) \leq 1 + f + g + 1 + g$$

implying $e \leq g + 1$, a contradiction. □

We use the notation introduced in Section 1: R^g and M^g denote the associated graded ring and module respectively.

Given an ideal I of R, let I^* denote the ideal of R^g generated by the initial forms of elements in I. Also, let r^* denote the initial form of an element r in R.

Theorem 2.5. *Let $(R, \mathfrak{m}, k)$ be a local ring with $\mathfrak{m}^4 = 0$ and balanced Hilbert series. If $a, b \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ form an exact pair of zero divisors then the following hold:*

(1) *their initial forms a^*, b^* in R^g form an exact pair of zero divisors;*
(2) *the graded rings $(R/aR)^\mathsf{g}$ and $R^\mathsf{g}/a^* R^\mathsf{g}$ are naturally isomorphic;*
(3) $\mathrm{H}_{R^\mathsf{g}/a^* R^\mathsf{g}}(t) = \mathrm{H}_{R/aR}(t) = \mathrm{H}_R(t)(1 + t)^{-1}$.

Proof. (1) Note that $(R^\mathsf{g}, \mathfrak{m}^\mathsf{g})$ is a local ring, where $\mathfrak{m}^\mathsf{g}$ denotes the irrelevant maximal ideal $R^\mathsf{g}_{\geqslant 1}$ of R^g. One has

$$\mu(a\mathfrak{m}) = \lambda(a\mathfrak{m}/a\mathfrak{m}^2) = \mu(a^* \mathfrak{m}^\mathsf{g}) \quad \text{and} \quad \mu(\mathfrak{m}) = \lambda(\mathfrak{m}/\mathfrak{m}^2) = \mu(\mathfrak{m}^\mathsf{g}).$$

Further, the property $a\mathfrak{m}^2 = \mathfrak{m}^3 = b\mathfrak{m}^2$ given by Proposition 2.3 implies

$$a^* (\mathfrak{m}^\mathsf{g})^2 = (\mathfrak{m}^\mathsf{g})^3 = b^* (\mathfrak{m}^\mathsf{g})^2.$$

Proposition 2.3 shows that a^*, b^* form an exact pair of zero divisors in R^g.

(2) The canonical homomorphism $R \to R/aR$ induces a surjective homomorphism of graded k-algebras $\varphi \colon R^\mathsf{g} \to (R/aR)^\mathsf{g}$. From [25, (1.1)], $\ker \varphi = a^* R^\mathsf{g}$ if and only if

(2.5.1) $\mathfrak{m}^i \cap aR = a\mathfrak{m}^{i-1} \quad \text{for all} \quad i \geq 1$.

The cases $i = 1$ and $i \geq 4$ are obvious and case $i = 2$ is given by Remark 2.2(1). Finally, $i = 3$ follows from Proposition 2.3(3) and we conclude that (2.5.1) holds.

(3) Note that $\mathrm{H}_{R^\mathsf{g}}(t) = \mathrm{H}_R(t)$ and $\mathrm{H}_{R^\mathsf{g}/a^* R^\mathsf{g}}(t) = \mathrm{H}_{(R/aR)^\mathsf{g}}(t) = \mathrm{H}_{R/aR}(t)$. The statement follows then from (1) and Lemma 2.1. □

3. Modules over short local rings

In this section we study the Koszul property of finite modules over local artinian rings $(R, \mathfrak{m}, k)$ with $\mathfrak{m}^4 = 0$ and balanced Hilbert series.

We establish our main result, Theorem 3.3, whose primary case of interest (over Gorenstein rings) is discussed in section 4. An important structural result is Proposition 3.9 yielding that, in the presence of special elements $a, b, c \in R$, the rings considered are homomorphic images of a codimension 2 complete intersection by means of a Golod homomorphism. This allows to apply results of Herzog and Iyengar [13, §5] on the Koszul property.

Proposition 3.1. *Let $(R, \mathfrak{m}, k)$ be a local ring with $\mathfrak{m}^4 = 0$ and balanced Hilbert series. If a, b in $\mathfrak{m} \smallsetminus \mathfrak{m}^2$ form an exact pair of zero divisors and M is an R/aR-module, then M is Koszul over R if and only if it is Koszul over R/aR.*

In particular, the ring R is Koszul if and only if the ring R/aR has this property.

Proof. Recall from 1.10 that a module is Koszul over a local ring if and only if its associated graded module has a linear resolution over the associated graded ring. From Theorem 2.5(1), the initial forms a^*, b^* form an exact pair of zero divisors in R^{g}. As M^{g} is a positively graded $R^{\mathsf{g}}/a^* R^{\mathsf{g}}$-module, Corollary 1.12 shows that M^{g} has a linear resolution over R^{g} precisely when M^{g} has a linear resolution over $R^{\mathsf{g}}/a^* R^{\mathsf{g}}$. This is the desired conclusion because the graded rings $R^{\mathsf{g}}/a^* R^{\mathsf{g}}$ and $(R/aR)^{\mathsf{g}}$ are naturally isomorphic by Theorem 2.5(2).   □

Recall that the nth *syzygy module* of a finite R-module M is the cokernel of the nth differential in a minimal free resolution of M over R. It is defined uniquely up to isomorphism and we denote it $\Omega_n^R(M)$. If an R-module N is isomorphic to $\Omega_n^R(M)$ for some $n \geq 0$, then we say that N is a *syzygy module of M*.

3.2. *Koszul syzygy modules.* If $(R, \mathfrak{m}, k)$ is a zero-dimensional local ring and M a finite R-module admitting a Koszul syzygy module then

$$\mathrm{H}_R(-t)\, \mathrm{P}_M^R(t) \in \mathbb{Z}[t].$$

This follows immediately from [13, Prop. 1.8].

In this way, if R is a zero-dimensional local ring over which every finite R-module M admits a Koszul syzygy module, then for each finite R-module M there exits a polynomial $p_M(t) \in \mathbb{Z}[t]$ satisfying

$$\mathrm{P}_M^R(t) = p_M(t)/\mathrm{H}_R(-t).$$

In general an uniform bound for the degree of $p_M(t)$ can not be derived. Indeed, if R is a self-injective non-regular Koszul ring, then for every $n \geq 0$ there exists a non-zero finite R-module M satisfying $k \cong \Omega_n^R(M)$. Since $\mathrm{P}_k^R(t) = \mathrm{H}_R(-t)^{-1}$ one then has

$$\mathrm{P}_M^R(t) = t^n\, \mathrm{P}_k^R(t) + q_M(t) = \frac{t^n}{\mathrm{H}_R(-t)} + q_M(t) = \frac{p_M(t)}{\mathrm{H}_R(-t)}$$

for some (uniquely determined) polynomials $q_M(t)$ and $p_M(t)$ of degrees $n-1$ and $n - 1 + \deg \mathrm{H}_R(t)$ respectively.

Our main result is as follows.

Theorem 3.3. *Let $(R, \mathfrak{m}, k)$ be a local ring with $\mathrm{H}_R(t) = 1 + et + et^2 + t^3$ and set $s = \mathrm{rank}_k(0 :_R \mathfrak{m})$. Assume there exists a non-zero exact zero divisor $a \in \mathfrak{m}$. The following statements are then equivalent:*

(1) R is Koszul.
(2) $e \geq s + 2$.

When they hold every finite R-module M has a syzygy module that is Koszul.

The proof of Theorem 3.3 is deduced from Theorem 3.11.

Corollary 3.4. *Let $(R, \mathfrak{m}, k)$ be a local ring with $\mathrm{H}_R(t) = 1 + et + et^2 + t^3$ and $e \geq 3$. Assume there exists a non-zero exact zero divisor $a \in \mathfrak{m}$ and R^{g} is quadratic. The ring R then is Koszul and every finite R-module M has a syzygy module that is Koszul.*

Proof. Let b be a complementary divisor for a, then a, b form an exact pair of zero divisors in $\mathfrak{m} \smallsetminus \mathfrak{m}^2$ by Lemma 2.4. Theorem 2.5(1) then establishes that the initial forms a^*, b^* of a, b form an exact pair of zero divisors in R^{g}. Further, $R^{\mathsf{g}}/a^* R^{\mathsf{g}}$ is

quadratic and $H_{R^{\mathrm{g}}/a^*R^{\mathrm{g}}}(t) = 1 + (e-1)t + t^2$ by Theorem 2.5(3). By [10, 2.12], it follows that

$$\operatorname{rank}_k(0 :_{R^{\mathrm{g}}} \mathfrak{m}^{\mathrm{g}}) \leq e-2$$

Applying Theorem 3.3 to the local ring R^{g}, we conclude that R^{g} is Koszul, hence R is Koszul as well. Thus R satisfies conditions (1)-(2) of the same theorem, concluding the proof. $\qquad\square$

Remark 3.5. Let R be an artinian graded k-algebra with irrelevant maximal ideal $\mathfrak{m}$. In particular, $(R, \mathfrak{m}, k)$ is a local ring. In [10], Conca, Rossi and Valla consider standard graded k-algebras R with $H_R(t) = 1 + et + et^2 + t^3$, where $e \geq 3$, that contain elements $l, m \in R_1$ with

(3.5.1) $\qquad lm = 0 \quad \text{and} \quad \operatorname{rank}_k(lR_1) = \operatorname{rank}_k(mR_1) = e-1\,.$

When R is quadratic or Gorenstein, Proposition 2.3 and the proof of [10, 2.13] show that a pair $l, m \in R_1$ satisfies (3.5.1) if and only if it is an exact pair of zero divisors. The Gorenstein case also follows from Proposition 4.1 in Section 4.

We introduce terminology to handle the technical ingredients of the proof.

3.6. *Conca generators.* Following [5], we say that an element $c \in \mathfrak{m}$ is a *Conca generator modulo aR* if it satisfies

$$\mathfrak{m}^2 + aR = c\mathfrak{m} + aR, \quad c \notin aR \quad \text{and} \quad c^2 \in aR.$$

Let $\bar{c}$ be the image of c in $\overline{R}$. These conditions imply $\overline{\mathfrak{m}}^3 = 0$ and $\bar{c} \notin \overline{\mathfrak{m}}^2 = \bar{c}\overline{\mathfrak{m}}$ by Nakayama's Lemma. Further, they are equivalent to

$$\mathfrak{m}^2 \subseteq c\mathfrak{m} + a\mathfrak{m}, \quad c \notin aR \quad \text{and} \quad c^2 \in a\mathfrak{m}$$

whenever a is part of an exact pair of zero divisors, as given by (2.5.1).

In the terminology of [5], the image in R/aR of a Conca generator modulo aR is exactly a Conca generator of the maximal ideal $\mathfrak{m}/aR$.

3.7. *Golod homomorphisms.* The notion of Golod homomorphism was introduced by Levin [16]. An epimorphism $\psi \colon (Q, \mathfrak{q}, k) \twoheadrightarrow (R, \mathfrak{m}, k)$ is said to be *Golod* if the following equality holds

(3.7.1) $\qquad P_k^R(t) = P_k^Q(t)(1 - t(P_R^Q(t) - 1))^{-1}.$

In the proof of the next proposition we use a result of Valabrega and Valla:

3.8. *Initial forms and regularity.* To establish the regularity of a sequence in a local ring Q it suffices to establish regularity of the corresponding sequence of initial forms in Q^{g}; see [25, 2.7].

As R is artinian, Cohen's Structure Theorem yields a regular local ring $(P, \mathfrak{p}, k)$ and a surjective local homomorphism

$$\pi \colon (P, \mathfrak{p}, k) \twoheadrightarrow (R, \mathfrak{m}, k)$$

such that $\ker \pi \subseteq \mathfrak{p}^2$.

Proposition 3.9. *Let $(R, \mathfrak{m}, k)$ be a local ring with $\mathfrak{m}^4 = 0$ and balanced Hilbert series. Let a, b, c be elements in $\mathfrak{m} \smallsetminus \mathfrak{m}^2$ such that a, b form an exact pair of zero divisors and c is a Conca generator modulo aR.*

The ring R then is Koszul, and there exist $u, v \in \mathfrak{p} \smallsetminus \mathfrak{p}^2$ and $w \in \mathfrak{p}^2$ such that uv, w is a P-regular sequence contained in $\ker \pi$ and such that the homomorphism

$$Q = P/(uv, w) \twoheadrightarrow R$$

induced by π is Golod.

Proof. Let $(\overline{R}, \overline{\mathfrak{m}})$ denotes the local ring $(R/aR, \mathfrak{m}/aR)$. For each $x \in R$ we let $\overline{x}$ denote the image of x in $\overline{R}$. The ring $\overline{R}$ is Koszul by [5, 1.1], so Proposition 3.1 shows that R is Koszul, and from (1.10.1) we obtain

(3.9.1) $$\mathrm{P}_k^R(t) = \mathrm{H}_R(-t)^{-1}.$$

The rest of the proof is conducted in three steps. The first one is to show that π factors through a map $\psi \colon Q \twoheadrightarrow R$ where Q is a complete intersection of the form $P/(uv, w)$ for some regular sequence uv, w in P such that $\pi(u) = a$.

We let $(\overline{P}, \overline{\mathfrak{p}})$ denote the regular local ring $(P/uP, \mathfrak{p}/uP)$ and for every $x \in P$ we denote $\overline{x}$ the image of x in $\overline{P}$.

In a second step we remark that the induced map $\overline{\pi} \colon \overline{P} \twoheadrightarrow \overline{R}$ is a Cohen presentation for $\overline{R}$ with $\ker \overline{\pi} \subseteq \overline{\mathfrak{p}}^2$ and maps a minimal generator for $\overline{\mathfrak{p}}$ to a Conca generator for $\overline{\mathfrak{m}}$. We then use [5, 1.4] to conclude that the map $\overline{\psi} \colon \overline{Q} \twoheadrightarrow \overline{R}$ induced by ψ on $\overline{Q} = Q/\tilde{u}Q$ is Golod, where $\tilde{u}$ denotes the image of u in Q.

The final step consists of establishing the equality (3.7.1) and therefore concluding that ψ itself is a Golod homomorphism.

Step 1. We show there exist elements u, v in $\mathfrak{p} \smallsetminus \mathfrak{p}^2$ and w in $\mathfrak{p}^2$ such that $\pi(u) = a$, $\pi(v) = b$ and uv, w forms a P-regular sequence. The map π then factors through a homomorphism

$$\psi \colon (Q, \mathfrak{q}, k) \twoheadrightarrow (R, \mathfrak{m}, k) \text{ where } Q = P/(uv, w).$$

Indeed, as c is a Conca generator modulo aR, there exists $d \in \mathfrak{m}$ such that

(3.9.2) $$c^2 = ad \quad \text{and} \quad \mathfrak{m}^2 + (a) = c\mathfrak{m} + (a).$$

One then has $\overline{c} \notin \overline{\mathfrak{m}}^2$ so a, c can be extended to a minimal generating set for $\mathfrak{m}$. There exists thus a minimal system of generators $x_1, \ldots, x_e$ for $\mathfrak{p}$ satisfying $\pi(x_1) = a$ and $\pi(x_2) = c$. Set $X_i = x_i{}^*$ in P^{g} and note that P^{g} is the polynomial ring over k on $X_1, \ldots, X_e$, since P is regular.

Pick $y \in \mathfrak{p}$ such that $\pi(y) = d$ and set

$$u = x_1 \quad \text{and} \quad w = x_2^2 - yx_1.$$

To carry out the first step we consider three cases.

Case 1. $\mathrm{rank}_k\{a^*, b^*\} = 1$.

We have thus $a^* R^{\mathsf{g}} = b^* R^{\mathsf{g}}$. We may then assume $b = a + f$ for some $f \in \mathfrak{m}^2$, since exact pairs of zero divisors are preserved up to multiplication by a unit in R. Choose $z \in \mathfrak{p}^2$ so that $\pi(z) = f$; the map π then factors through the map

(3.9.3) $$Q = \frac{P}{(x_1(x_1 + z), x_2{}^2 - yx_1)} \xrightarrow{\ \psi\ } R.$$

Set $v = x_1 + z$. We claim that the sequence uv, w is P-regular. In P^{g} one has

$$(uv)^* = X_1{}^2 \quad \text{and} \quad w^* = \begin{cases} X_2{}^2 - y^* X_1, & \text{if } y \in \mathfrak{p} \smallsetminus \mathfrak{p}^2 \\ X_2{}^2, & \text{if } y \in \mathfrak{p}^2 \end{cases} .$$

Since both sequences

$$X_1{}^2, X_2{}^2 \quad \text{and} \quad X_1{}^2,\ X_2{}^2 - y^* X_1$$

are $k[X_1, \ldots, X_e]$-regular, we conclude from 3.8 that uv, w is P-regular.

Case 2. $\operatorname{rank}_k \{a^*, b^*, c^*\} = 3$.

As a, b, c can be extended to a minimal generating set for $\mathfrak{m}$, then we may assume $\pi(x_3) = b$. The map π then factors through the homomorphism

$$(3.9.4) \qquad\qquad Q = \frac{P}{(x_1 x_3, x_2{}^2 - y x_1)} \xrightarrow{\ \psi\ } R.$$

Set $v = x_3$. We claim that the sequence uv, w is P-regular. In P^{g} one has

$$(uv)^* = X_1 X_3 \quad \text{and} \quad w^* = \begin{cases} X_2{}^2 - y^* X_1, & \text{if } y \in \mathfrak{p} \smallsetminus \mathfrak{p}^2 \\ X_2{}^2, & \text{if } y \in \mathfrak{p}^2 \end{cases}.$$

Since both sequences

$$X_1 X_3, X_2{}^2 \quad \text{and} \quad X_1 X_3,\ X_2{}^2 - y^* X_1$$

are $k[X_1, \ldots, X_e]$-regular, we conclude from 3.8 that uv, w is P-regular.

Case 3. $\operatorname{rank}_k \{a^*, b^*, c^*\} = 2 = \operatorname{rank}_k \{a^*, b^*\}$.

In other terms, $\bar{c} = \bar{b} + \bar{g}$ in R/aR for some $g \in \mathfrak{m}^2$. Since $\overline{\mathfrak{m}}^3 = \overline{a \mathfrak{m}^2} = 0$, we have

$$\bar{b}^2 = \bar{c}^2 = 0 \quad \text{and} \quad \overline{\mathfrak{m}}^2 = \overline{\mathfrak{m}}\,\bar{c} = \overline{\mathfrak{m}}\,\bar{b}.$$

Thus we may replace c with b and then (3.9.2) yields $b^2 = ad$.

If $d \in bR$, then $b^2 = ad = 0$ which implies $b \in (0 :_R b) = aR$. As $b \notin \mathfrak{m}^2$, it would then fall into *Case 1*. It remains to treat the case when $d \notin bR$.

If $d \notin bR$, the map π then factors through the homomorphism

$$(3.9.5) \qquad\qquad Q = \frac{P}{(x_1 x_2, x_2{}^2 - y x_1)} \xrightarrow{\ \psi\ } R.$$

Set $v = x_2$. The sequence uv, w is P-regular: because P is a UFD, x_1, x_2 are irreducible elements and $x_2 \nmid y$.

Step 2. We observe that the map $\overline{\psi} \colon \overline{Q} \twoheadrightarrow \overline{R}$ induced by $\psi \colon Q \twoheadrightarrow R$ on $\overline{Q} = Q/\tilde{u}Q$ is Golod.

By choice, $u = x_1$ is not in $\mathfrak{p}^2$, so the ring $\overline{P} = P/uP$ is regular. The induced map $\overline{\pi} \colon \overline{P} \twoheadrightarrow \overline{R}$ is a Cohen presentation with $\ker \overline{\pi} \subseteq \overline{\mathfrak{p}}^2$ such that the element $\overline{x}_2 \in \overline{\mathfrak{p}} \smallsetminus \overline{\mathfrak{p}}^2$ maps to the Conca generator $\bar{c}$ of $\overline{\mathfrak{m}}$. Note that there are canonical isomorphisms

$$(3.9.6) \qquad\qquad \overline{Q} = \frac{Q}{\tilde{u}Q} \cong \frac{P}{(u, w)} \cong \frac{\overline{P}}{\overline{x}_2^2\, \overline{P}}.$$

Under this identification the map $\overline{\pi} \colon \overline{P} \twoheadrightarrow \overline{R}$ factors through $\overline{\psi} \colon \overline{Q} \twoheadrightarrow \overline{R}$, and it follows from [5, 1.4] that $\overline{\psi}$ is a Golod homomorphism.

Step 3. We now prove that the homomorphism $\psi \colon Q \twoheadrightarrow R$ is Golod.

We first note that, from Remark 1.4, the images $\tilde{u}, \tilde{v}$ of u, v in Q form an exact pair of zero divisors. Thus, as in Remark 1.1, $\mathbf{F}_{\tilde{u},\tilde{v}}(Q)$ is a minimal free resolution of $\overline{Q}$ over Q. Since $\psi(\tilde{u}) = a$ and $\psi(\tilde{v}) = b$, one has an isomorphism of complexes

$$\mathbf{F}_{\tilde{u},\tilde{v}}(Q) \otimes_Q R \cong \mathbf{F}_{a,b}(R),$$

which induces isomorphisms

$$\mathrm{Tor}_n^Q(\overline{Q}, R) = \mathrm{H}_n(\mathbf{F}_{\tilde{u},\tilde{v}}(Q) \otimes_Q R) = \mathrm{H}_n(\mathbf{F}_{a,b}(R)) = 0, \quad \text{for all } n > 0.$$

Therefore,

$$(3.9.7) \qquad\qquad\qquad P_{\overline{R}}^{\overline{Q}}(t) = P_R^Q(t).$$

Since $\overline{P}$ is a regular local ring, (3.9.6) and [24, Thm 6] yield equalities

$$(3.9.8) \qquad\qquad\qquad P_k^Q(t) = (1+t)^e(1-t^2)^{-2},$$

$$(3.9.9) \qquad\qquad\qquad P_k^{\overline{Q}}(t) = (1+t)^{e-1}(1-t^2)^{-1},$$

where $e = \mu(\mathfrak{m}) = \mu(\overline{\mathfrak{m}}) + 1$.

Now we can write the following sequence of equalities

$$\mathrm{P}_k^R(t) = \frac{\mathrm{P}_k^{\overline{Q}}(t)}{(1 - t(P_{\overline{R}}^{\overline{Q}}(t) - 1))(1-t)} = \frac{(1+t)^{e-1}}{(1 - t(P_{\overline{R}}^{\overline{Q}}(t) - 1))(1-t^2)(1-t)}$$

$$= \frac{(1+t)^e}{(1 - t(P_R^Q(t) - 1))(1-t^2)^2} = \frac{P_k^Q(t)}{1 - t(P_R^Q(t) - 1)}$$

The first equality above follows from Step 2 and Theorem 1.7, the rest from (3.9.9), (3.9.7) and (3.9.8) respectively. The resulting equality yields that the map $\psi\colon Q \twoheadrightarrow R$ is Golod. $\qquad\square$

Remark 3.10. Let $\varphi\colon (R, \mathfrak{m}, k) \hookrightarrow (R', \mathfrak{m}', k')$ be an *inflation*; that is, a flat homomorphism of local rings such that $\varphi(\mathfrak{m}) = \mathfrak{m}'$. One then has $\mathrm{H}_R(t) = \mathrm{H}_{R'}(t)$.

For every local ring $(R, \mathfrak{m}, k)$ there exits a local ring $(R', \mathfrak{m}', k')$ and an inflation $R \hookrightarrow R'$ where k' is an algebraically closed field k, see [9, AC IX.41].

An inflation $\overline{\varphi}\colon (\overline{R}, \overline{\mathfrak{m}}, k) \hookrightarrow (\overline{R}', \overline{\mathfrak{m}}', k')$, where $\overline{R} = R/aR$, $\overline{\mathfrak{m}} = \mathfrak{m}/aR$ and $(\overline{R}', \overline{\mathfrak{m}}', k')$ is a local ring, lifts to an inflation $\varphi\colon (R, \mathfrak{m}, k) \hookrightarrow (R', \mathfrak{m}', k')$ satisfying $R'/\varphi(a)R' \cong \overline{R}'$, see [9, AC IX.40].

Given a minimal free resolution $\mathbf{F}$ of an R-module M, the complex $\mathbf{F} \otimes_R R'$ is a minimal free resolution of the R'-module $M' = M \otimes_R R'$. Thus one gets

$$\Omega_n^{R'}(M') \cong \Omega_n^R(M) \otimes_R R' \quad \text{and} \quad \mathrm{P}_{M'}^{R'}(t) = \mathrm{P}_M^R(t).$$

Further, as noted in [5, 1.8], the extended module M' is Koszul over R' precisely when M is Koszul over R.

Theorem 3.11. *Let $(R, \mathfrak{m}, k)$ be a local ring with $\mathfrak{m}^4 = 0$ and $\mathrm{H}_R(-1) = 0$. Assume that there exist an inflation $(R, \mathfrak{m}, k) \to (R', \mathfrak{m}', k')$ and elements a, b, c in $\mathfrak{m}' \smallsetminus \mathfrak{m}'^2$ such that a, b form an exact pair of zero divisors in R' and c is a Conca generator modulo aR'.*

The ring R then is Koszul and every finite R-module M has a syzygy module that is Koszul. In particular, $\mathrm{P}_M^R(t)\,\mathrm{H}_R(-t)^{-1} \in \mathbb{Z}[t]$.

Proof. In view of Remark 3.10 and 1.3, we can replace $(R', \mathfrak{m}', k')$ with $(R, \mathfrak{m}, k)$ and $M \otimes_R R'$ with M. The ring R then satisfies the hypothesis of Proposition 3.9, and hence is Koszul.

Proposition 3.9 and [13, 5.9] show that every finite R-module M has a syzygy module that is Koszul over R; hence $\mathrm{P}_M^R(t)\,\mathrm{H}_R(-t)^{-1} \in \mathbb{Z}[t]$ from 3.2. $\qquad\square$

Proof of Theorem 3.3. Let b be a complementary divisor for a. From Lemma 2.4(2) a, b form an exact pair of zero divisors in $\mathfrak{m} \smallsetminus \mathfrak{m}^2$, since $\mu(\mathfrak{m}^3) = 1 = 3 - 2 \leq e - 2$.

Recall that $\overline{R} = R/aR$ is a local ring with maximal ideal $\overline{\mathfrak{m}} = \mathfrak{m}/aR$ and $\mathrm{H}_{\overline{R}}(t) = 1 + (e-1)t + t^2$ as given by Theorem 2.5(3), then

$$\mathrm{rank}_k(\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2) = e - 1 \quad \text{and} \quad \mathrm{rank}_k(\overline{\mathfrak{m}}^2) = 1.$$

Lemma 1.5 yields

$$\mathrm{rank}_k(0 :_{\overline{R}} \overline{\mathfrak{m}}) = s$$

and [5, 4.1] shows that $\overline{R}$ is Koszul if and only if

$$\mathrm{rank}_k(0 :_{\overline{R}} \overline{\mathfrak{m}}) \leq \mathrm{rank}_k(\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2) - 1$$

that is, $s \leq e - 2$. Proposition 3.1 gives that R is Koszul if and only if $\overline{R}$ is Koszul, establishing thus the equivalence of (1) and (2).

Assuming that (1) and (2) hold, then [5, Thm 4.1,(v) $\Longrightarrow$ (i)] and Remark 3.10 establish the existence of an inflation $\varphi \colon (R, \mathfrak{m}, k) \hookrightarrow (R', \mathfrak{m}', k')$ such that $\mathfrak{m}' \smallsetminus \mathfrak{m}'^2$ contains an element c that is a Conca generator modulo $\varphi(a)R'$. Finally, from 1.3 we can apply Theorem 3.11 concluding the proof. $\qquad\square$

4. Short local Gorenstein rings

In this section we study exact pairs of zero divisors in local Gorenstein rings $(R, \mathfrak{m}, k)$ with $\mathfrak{m}^4 = 0$. We prove the Gorenstein version of our main result as presented in the introduction.

Recall that R is Gorenstein if and only if $\overline{R} = R/aR$ is Gorenstein, as discussed in Remark 1.6.

Proposition 4.1. *Let* $(R, \mathfrak{m}, k)$ *be a Gorenstein local ring satisfying* $\mathfrak{m}^4 = 0$ *and* $\mu(\mathfrak{m}) = e \geq 3$. *Given* $0 \neq a \in \mathfrak{m}$ *the following statements are equivalent:*

(1) $(0 :_R a)$ *is a principal ideal;*
(2) a *is an exact zero divisor;*
(3) a *is in* $\mathfrak{m} \smallsetminus \mathfrak{m}^2$ *and there exists* $b \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ *such that the following equivalent conditions hold:*
   (3.1) a, b *form an exact pair of zero divisors;*
   (3.2) $ab = 0$ *and* $\mu(a\mathfrak{m}) = e - 1 = \mu(b\mathfrak{m})$.

If any of the conditions (1)-(3) *holds, then* $\mathrm{H}_R(t) = 1 + et + et^2 + t^3$.

Proof. We first show that given $a, b \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$, (3.1) implies $\mathrm{H}_R(t) = 1 + et + et^2 + t^3$. Set $(\overline{R}, \overline{\mathfrak{m}}) = (R/aR, \mathfrak{m}/aR)$. The hypothesis that $a \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ implies $\mu(\overline{\mathfrak{m}}) = e - 1$. Since R is Gorenstein, Remark 1.6 shows that $\overline{R}$ is Gorenstein. Note that

(4.1.1) $$\overline{\mathfrak{m}}^2 = (\mathfrak{m}^2, aR)/aR \cong \mathfrak{m}^2/aR \cap \mathfrak{m}^2 = \mathfrak{m}^2/a\mathfrak{m}$$

where the equality $aR \cap \mathfrak{m}^2 = a\mathfrak{m}$ is given by Remark 2.2(1).

Assume $\overline{\mathfrak{m}}^2 = 0$. Since $\overline{R}$ is Gorenstein, it follows that $\overline{\mathfrak{m}} = 0$ or $\mu(\overline{\mathfrak{m}}) = 1$, hence $e = 1$ or $e = 2$. We may then assume $\overline{\mathfrak{m}}^2 \neq 0$. If $\mathfrak{m}^3 = 0$, then, since R is Gorenstein, $\mathfrak{m}^2 \neq 0$ and $a \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$, it follows that $\mathfrak{m}^2 = a\mathfrak{m}$ and thus $\overline{\mathfrak{m}}^2 = 0$, a contradiction.

We have thus $\mathfrak{m}^3 \neq 0$. Since R is Gorenstein with $\mathfrak{m}^4 = 0$, we have $\mathfrak{m}^3 = (0 :_R \mathfrak{m})$ and $\mu(\mathfrak{m}^3) = 1$. Remark 2.2(4) gives

$$(4.1.2) \qquad\qquad \lambda(R) = \mu(\mathfrak{m}^2) + e + 2.$$

The inclusion $a\mathfrak{m}^2 \subseteq \mathfrak{m}^3$ implies $a\mathfrak{m}^2 = \mathfrak{m}^3$ or $a\mathfrak{m}^2 = 0$. If $a\mathfrak{m}^2 = 0$, then $a\mathfrak{m} \subseteq (0 :_R \mathfrak{m})$. Since $a \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$, we have $a\mathfrak{m} \neq 0$, hence $a\mathfrak{m} = (0 :_R \mathfrak{m}) = \mathfrak{m}^3$.

We have thus one of the two cases:

(i) $a\mathfrak{m} = \mathfrak{m}^3$ and $a\mathfrak{m}^2 = 0$, hence $\mu(a\mathfrak{m}) = 1$ and $\mu(a\mathfrak{m}^2) = 0$.
(ii) $a\mathfrak{m}^2 = \mathfrak{m}^3$, hence $\mu(a\mathfrak{m}^2) = 1$.

In either case, $\mathfrak{m}^3 \subseteq a\mathfrak{m}$, hence $\overline{\mathfrak{m}}^3 = 0 \neq \overline{\mathfrak{m}}^2$. Since $\overline{R}$ is Gorenstein, we conclude that $\mathrm{rank}_k(\overline{\mathfrak{m}}^2) = 1$, hence $\mathrm{rank}_k(\mathfrak{m}^2/a\mathfrak{m}) = 1$ by (4.1.1). A length count in the exact sequence

$$0 \to a\mathfrak{m}/\mathfrak{m}^3 \to \mathfrak{m}^2/\mathfrak{m}^3 \to \mathfrak{m}^2/a\mathfrak{m} \to 0$$

gives

$$(4.1.3) \qquad\qquad \mathrm{rank}_k(a\mathfrak{m}/\mathfrak{m}^3) = \mu(\mathfrak{m}^2) - 1$$

If (i) holds, then $a\mathfrak{m}/\mathfrak{m}^3 = 0$, hence $\mu(\mathfrak{m}^2) = 1$. Remark 2.2 gives

$$\lambda(aR) = \mu(a\mathfrak{m}) + \mu(a\mathfrak{m}^2) + 1 = 2 = \mu(\mathfrak{m}^2) + 1$$

If (ii) holds, then $\mu(a\mathfrak{m}) = \mu(\mathfrak{m}^2) - 1$ by (4.1.3) and Remark 2.2 gives:

$$\lambda(aR) = \mu(a\mathfrak{m}) + \mu(a\mathfrak{m}^2) + 1 = \mu(\mathfrak{m}^2) + 1$$

In either case, Remark 2.2 and (4.1.2) yield

$$\lambda(0 : a) = \lambda(R) - \lambda(aR) = \big(\mu(\mathfrak{m}^2) + e + 2\big) - (\mu(\mathfrak{m}^2) + 1) = e + 1$$

Similar formulas hold for $\lambda(bR)$ and $\lambda(0 : b)$, hence

$$\mu(\mathfrak{m}^2) + 1 = \lambda(aR) = \lambda(0 : b) = e + 1$$

It follows that $\mu(\mathfrak{m}^2) = e$, thus $\mathrm{H}_R(t) = 1 + et + et^2 + t^3$.

We now establish the equivalence of conditions (3.1) and (3.2) for a, b in $\mathfrak{m} \smallsetminus \mathfrak{m}^2$. If (3.1) holds, then $\mathrm{H}_R(-1) = 0$ by the above argument. The equalities $\mu(a\mathfrak{m}) = e - 1 = \mu(b\mathfrak{m})$ then follow from Proposition 2.3. Conversely, assume $ab = 0$ and $\mu(a\mathfrak{m}) = e - 1 = \mu(b\mathfrak{m})$. If $a\mathfrak{m} = 0$ then $\mu(a\mathfrak{m}) = e - 1 = 0$, contradicting the hypothesis $e \geq 3$. If $\mathfrak{m}^3 = 0$ then, as $a\mathfrak{m} \neq 0$ and $\lambda(0 :_R \mathfrak{m}) = 1$, one has $a\mathfrak{m} = \mathfrak{m}^2 = (0 :_R \mathfrak{m})$ and $\mu(a\mathfrak{m}) = 1$. The equality $\mu(a\mathfrak{m}) = e - 1$ gives $e = 2$, contradicting the hypothesis.

We have thus $\mathfrak{m}^3 \neq 0$. Since R is Gorenstein, one has $(0 : \mathfrak{m}) = \mathfrak{m}^3$ and $\mu(\mathfrak{m}^3) = 1$. Note that $a\mathfrak{m}^2 \subseteq \mathfrak{m}^3$ hence we have $a\mathfrak{m}^2 = 0$ or $a\mathfrak{m}^2 = \mathfrak{m}^3$. We will show that that $a\mathfrak{m}^2 \neq 0$, implying that $a\mathfrak{m}^2 = \mathfrak{m}^3$. Similarly, $b\mathfrak{m}^2 = \mathfrak{m}^3$ and Proposition 2.3 shows that a, b is an exact pair of zero divisors.

Indeed, assume $a\mathfrak{m}^2 = 0$. We then have $a\mathfrak{m} \subseteq (0 : \mathfrak{m})$, hence $a\mathfrak{m} \subseteq \mathfrak{m}^3$. If $a\mathfrak{m} = 0$, then $a \in (0 : \mathfrak{m}) = \mathfrak{m}^3$, a contradiction. It remains thus that $a\mathfrak{m} = \mathfrak{m}^3$, and hence $\mu(a\mathfrak{m}) = 1$. The equality $\mu(a\mathfrak{m}) = e - 1$ then yields $e = 2$, a contradiction.

To establish the equivalence of (1) and (2) it is enough to recall that, since R is Gorenstein, one has $(0 :_R (0 :_R I)) = I$ for every ideal I.

It then remains to establish that (2) implies (3) (i), since the reverse implication is clear. For this, it is enough to show that a, b are in $\mathfrak{m} \smallsetminus \mathfrak{m}^2$. Note that $0 \neq a \in \mathfrak{m}$ implies $0 \neq b \in \mathfrak{m}$. As R is Gorenstein $\mu(\mathfrak{m}^3) \leq \mathrm{rank}_k(0 :_R \mathfrak{m}) = 1$, thus Lemma 2.4 (2) yields $a, b \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$, concluding the equivalence of (2) and (3). $\qquad\square$

Theorem 4.2. *Let $(R, \mathfrak{m}, k)$ be a local Gorenstein ring with $\mu(\mathfrak{m}) \geq 3$ and $\mathfrak{m}^4 = 0$.*

If there exists a non-zero element $a \in \mathfrak{m}$ such that the ideal $(0 :_R a)$ is principal, then the following hold:

(1) *R^{g} is Gorenstein;*

(2) *R is Koszul;*

(3) *every finite R-module M has a syzygy module that is Koszul.*

Proof. Proposition 4.1 shows that $\mathrm{H}_R(t) = 1 + et + et^2 + t^3$ with $e = \mu(\mathfrak{m})$ and yields the existence of a element b such that $a, b \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ form an exact pair of zero divisors.

From Remark 1.6, $\overline{R} = R/aR$ is Gorenstein with $\overline{\mathfrak{m}}^3 = 0$, thus $(0 :_R \overline{\mathfrak{m}}) = \overline{\mathfrak{m}}^2$ and $\mu(\overline{\mathfrak{m}}^2) = 1$. It follows that $(0 :_{R^{\mathsf{g}}} \overline{\mathfrak{m}}^{\mathsf{g}}) = \overline{\mathfrak{m}}^{\mathsf{g}\,2}$ and $\mu(\overline{\mathfrak{m}}^{\mathsf{g}\,2}) = 1$, hence $\overline{R}^{\mathsf{g}}$ is Gorenstein as well. From Theorem 2.5(2) and (1), $R^{\mathsf{g}}/a^* R^{\mathsf{g}}$ is thus a Gorenstein local ring where a^* is an exact zero divisor. It then follows from Remark 1.6 that R^{g} is Gorenstein, establishing (1).

Since $\mathrm{rank}_k(0 :_R \mathfrak{m}) = 1$, the hypothesis $\mu(\mathfrak{m}) \geq 3$ establishes condition (2) of Theorem 3.3 concluding (2) and (3). $\qquad\qquad\square$

Remark 4.3. The set of standard graded Gorenstein k-algebras R with $\mathrm{H}_R(t) = 1 + et + et^2 + t^3$ is in bijective correspondence with the set of degree 3 forms (up to scalars) in e variables over k, via the 'inverse systems' correspondence of Macaulay, as described in [10, §6]. As a consequence, such algebras can be parametrized by means of a projective space $\mathbf{P}(A_3)$, where A_3 is the degree three graded component of a polynomial ring over k in e variables. In view of Remark 3.5, the proof of [10, 6.4] shows that when $e \geq 3$ there exists a non-empty Zariski open subset of $\mathbf{P}(A_3)$, whose points correspond to Gorenstein standard graded k-algebras admitting an exact pair of zero divisors (cf [10, 6.5]). In other words: a generic Gorenstein standard graded k-algebra of socle degree 3 has an exact pair of zero divisors in degree one.

We now derive the numerical version of Theorem 4.2 as presented in the introduction.

Corollary 4.4. *Let $(R, \mathfrak{m}, k)$ be Gorenstein with $\mathfrak{m}^4 = 0$ and $\mu(\mathfrak{m}) = e \geq 3$.*

Assume there exist a non-zero element $a \in \mathfrak{m}$ such that the ideal $(0 :_R a)$ is principal then the following hold:

(1) *$\mathrm{H}_R(t) = 1 + et + et^2 + t^3$;*

(2) *$\mathrm{H}_R(-t)\,\mathrm{P}_k^R(t) = 1$;*

(3) *$\mathrm{H}_R(-t)\,\mathrm{P}_M^R(t)$ is in $\mathbb{Z}[t]$, for every finite R-module M.*

Proof. (1) It follows directly from Proposition 4.1 that $\mathrm{H}_R(t) = 1 + et + et^2 + t^3$.

(2) and (3) follow from the structural conditions (2) and (3) in Theorem 4.2 together with (1.10.1) and 3.2 respectively. $\qquad\qquad\square$

To finish, we discuss the condition on $\mu(\mathfrak{m})$ in the theorem. Any Gorenstein ring with $\mu(\mathfrak{m}) \leq 2$ is a complete intersection, see [21, Prop. 5], [6, §5], and Poincaré series over such rings are well understood.

4.5. *Artinian complete intersections.* Let $(R, \mathfrak{m}, k)$ be an Artinian complete intersection ring, M be a finite module over R and let $\epsilon(R)$ denote the Hilbert-Samuel multiplicity of R. One always has:

$$(4.5.1) \qquad \epsilon(R) \geq 2^{\mu(\mathfrak{m})}, \ \mathrm{P}_k^R(t) = (1-t)^{-\mu(\mathfrak{m})} \ \text{ and } \ (1-t^2)^{\mu(\mathfrak{m})}\,\mathrm{P}_M^R(t) \in \mathbb{Z}[t],$$

(cf [8, §7, Prop. 7], [24, Thm. 6] and [12, 4.1]).

Furthermore, the following conditions are equivalent:

(1) R has *minimal multiplicity, i.e.* $\epsilon(R) = 2^{\mu(\mathfrak{m})}$,

(2) R is Koszul,

(3) $\mathrm{H}_R(t) = (1 + t)^{\mu(\mathfrak{m})}$;

and when these conditions hold, one has

$$(4.5.2) \qquad\qquad (1 - t)^{\mu(\mathfrak{m})} \, \mathrm{P}_M^R(t) \in \mathbb{Z}[t].$$

Indeed, Avramov shows in [2, (2.3)] that (1) implies (2) and (4.5.2). The implication (2) $\Rightarrow$ (3) follows from (1.10.1) and (4.5.1). On the other hand one has $\epsilon(R) = \lambda(R) = \mathrm{H}_R(1)$ as R is Artinian, so it becomes clear that (3) $\Rightarrow$ (1).

Acknowledgments

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