# SHORT KOSZUL MODULES

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*To Ralf Fröberg, on his 65th birthday*

Abstract. This article is concerned with graded modules $M$ with linear resolutions over a standard graded algebra $R$. It is proved that if such an $M$ has Hilbert series $H_M(s)$ of the form $ps^d + qs^{d+1}$, then the algebra $R$ is Koszul; if, in addition, $M$ has constant Betti numbers, then $H_R(s) = 1 + es + (e-1)s^2$. When $H_R(s) = 1+es+rs^2$ with $r \le e-1$, and $R$ is Gorenstein or $e = r+1 \le 3$, it is proved that generic $R$-modules with $q \le (e-1)p$ are linear.

## Introduction

We study homological properties of graded modules over a standard graded commutative algebra $R$ over a field $k$; recall that this means that $R_0$ equals $k$ and $R$ is generated over $k$ by finitely many elements of degree one.

Unless $R$ is a polynomial ring, any general statement about $R$-modules necessarily concerns modules of infinite projective dimension. Various attractive conjectures have been based on expectations that homological properties of modules of finite projective dimension extend—in appropriate form—to all modules.

It is remarkable that several such conjectures have been refuted by using modules $M$, whose infinite minimal free resolution display the simplest numerical pattern: the graded Betti numbers $\beta^R_{i,j}(M)$ are zero for all $j \neq i$ (that is to say, $M$ is *Koszul*), and $\beta^R_{i,i}(M) = p$ for some $p \ge 1$ and all $i \ge 0$; see [11, 15, 16]. Furthermore, in those examples both $R$ and $M$ have special properties: $R$ is a *Koszul algebra*, meaning that $k$ is a Koszul module, the Hilbert series $H_R(s) = \sum_{j\in\mathbb{Z}} \operatorname{rank}_k R_j s^j$ has the form $1 + es + (e-1)s^2$, and one has $H_M(s) = p + (e-1)ps$.

This is a striking amalgamation of structural and numerical restrictions. The following result, extracted from Theorems 1.6 and 4.1(1), shows that it is inevitable.

**Theorem 1.** *Let $R$ be a standard graded algebra and $M$ a non-zero $R$-module.*
*If $M_j = 0$ for $j \neq 0, 1$ and $M$ is Koszul, then $R$ is a Koszul algebra.*
*If, furthermore, $\beta^R_{i,i}(M) = p$ for some $p$ and all $i \ge 0$, then*

$$H_R(s) = (1+s)\cdot(1+(e-1)s) \quad \textit{and} \quad H_M(s) = p\cdot(1+(e-1)s)\,.$$

The main themes of the paper are to find conditions when such modules actually exist, and to establish whether they display some "generic" behavior. An important step is to identify a set-up where similar questions may be stated in meaningful terms and answers can be tested against existing examples.

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Much of the discussion is carried out in the broader framework of Koszul modules over Koszul algebras. Conca, Trung, and Valla [7] proved if $R$ is a Koszul algebra with $H_R(s) = 1+es+rs^2$, then $e^2 \geq 4r$ holds, and that *generic* quadratic algebras $R$ satisfying this inequality are Koszul.

To analyze the restrictions imposed on $M$ by Theorem 1, we fix a Koszul algebra $R$ with $H_R(s) = 1 + es + rs^2$ and use *multiplication tables* to parametrize the $R$-modules with underlying vector space $k^p \oplus k^q(-1)$; see Section 2. This identifies such modules with the points of the affine space $\mathrm{M}_{ep\times q}(k)$ of $ep \times q$ matrices with elements in $k$, equipped with the Zariski topology.

We study the following questions concerning the subset $\boldsymbol{L}_{p,q}(R) \subseteq \mathrm{M}_{ep\times q}(k)$ corresponding to Koszul $R$-modules: *When is $\boldsymbol{L}_{p,q}(R)$ non-empty? When is its interior non-empty?* Recall that, in a topological space, the *interior* of a subset $X$ is the largest open set contained in $X$; in $\mathrm{M}_{ep\times q}(k)$ every subset with non-empty interior is *dense*, because affine spaces are irreducible.

It is not hard to show that $2q \leq (e+\sqrt{e^2-4r})p$ is a necessary condition for $\boldsymbol{L}_{p,q}$ to be non-empty; see Corollary 1.7. To establish sufficient conditions, we assume that $e \geq r+1$ holds. Conca [5] proved that, generically, algebras $R$ satisfying this inequality contain an element $x \in R_1$ with $x^2 = 0$ and $xR_1 = R_2$. In an earlier paper, [2], we called such an $x$ a *Conca generator* of $R$ and demonstrated that the existence of one impacts the structure of the minimal free resolutions of every $R$-module. The results of [2] are widely used here.

The following statement is condensed from Propositions 5.4, 5.5, and 5.6. Its proof depends on the study, in Section 3, of the loci $\boldsymbol{L}^m_{p,q}(R)$ of modules whose minimal free resolution is linear for the first $m$ steps.

**Theorem 2.** *Let $R$ be a standard graded algebra with a Conca generator.*

*For $p, q \in \mathbb{N}$ the linear locus $\boldsymbol{L}_{p,q}(R)$ of $\mathrm{M}_{ep\times q}(k)$ is not empty when $q \leq (e-1)p$, and has a non-empty interior when $q \leq \max\{e-1\,,\,(e-r)p\}$.*

In the motivating case when $H_R(s) = 1 + es + (e-1)s^2$, Theorem 2 shows that *generically* $\boldsymbol{L}_{p,(e-1)p}(R)$ is not empty. Computer experiments suggest that even its interior may be non-empty. Indeed, letting $R$ be a quotient of $k[x_1,\dots,x_e]$ by $\binom{e+1}{2} - (e-1)$ "random" quadratic forms and $M$ an $R$-module presented by a "random" $p\times p$ matrix of linear forms in $x_1,\dots,x_e$ , one gets $\beta^R_{i,j}(M) = 0$ for $j \neq i$ and $\beta^R_{i,i}(M) = p$ with unsettling frequency and for "large" values of $i$.

In the next theorem, contained in Propositions 6.3 and 6.4, we describe algebras with non-empty open sets of linear modules, under mild hypotheses on $k$.

**Theorem 3.** *Let $R$ be a short standard graded $k$-algebra.*

*If $R$ is Gorenstein, then for all pairs $(p, q)$ with $p \geq 1$ the set $\boldsymbol{L}_{p,q}(R)$ is open in $\mathrm{M}_{ep\times q}(k)$; it is not empty when $q \leq (e-1)p$ and there exists a non-zero element $x \in R_1$ with $x^2 = 0$ (in particular, when $k$ is algebraically closed).*

*If $R$ is quadratic with $H_R(s) = 1 + es + (e-1)s^2$ and $e \leq 3$, then for all $p \geq 1$ the set $\boldsymbol{L}_{p,(e-1)p}(R)$ is open in $\mathrm{M}_{ep\times(e-1)p}(k)$, and is not empty if $k$ is infinite.*

For $R$ as in the last statement of Theorem 3, the $R$-modules in $\boldsymbol{L}_{p,(e-1)p}(R)$ are described as those that are *periodic of period* 2, see Section 4. Over rings with $e \geq 4$, these classes may be distinct, and new ones appear; see [11, 16].

The generic behavior of Koszul modules with constant Betti numbers over a generic Koszul algebra $R$ with $H_R(s) = 1 + 4s + 3s^2$ still is a mystery.

## Notation

Let $(R, \mathfrak{m}, k)$ *be a graded algebra*; in this paper, the phrase introduces the following hypotheses and notation: $k$ is a field, $R = \oplus_{j\in\mathbb{Z}} R_j$ is a commutative graded $k$-algebra finitely generated over $R_0 = k$, $R_j = 0$ for $j < 0$, and $\mathfrak{m} = \oplus_{j\geqslant 1} R_j$.

Let $M = \oplus_{j\in\mathbb{Z}} M_j$ be a graded $R$-module, here always assumed finite. For every $d \in \mathbb{Z}$, we let $M(d)$ denote the graded $R$-modules $M(d)_j = M_{j+d}$ for each $j$. Set:

$$\inf M = \inf\{j \in \mathbb{Z} \mid M_j \neq 0\}\,,$$

$$H_M(s) = \sum_{j=\inf M}^{\infty} (\operatorname{rank}_k M_j) s^j \in \mathbb{Z}(\!(s)\!)\,.$$

The formal Laurent series above is the *Hilbert series* of $M$.

It is implicitly assumed that homomorphisms of graded $R$-modules preserve degrees. In this category, the free modules are isomorphic to direct sums of copies of $R(d)$, with various $d$. Every graded $R$-module $M$ has a *minimal free resolution*

$$F = \quad \cdots \to F_n \xrightarrow{\partial_n} F_{n-1} \to \cdots \to F_1 \xrightarrow{\partial_1} F_0 \to 0$$

with each $F_n$ finite free and $\partial_n(F_n) \subseteq \mathfrak{m}F_{n-1}$. Computing with it, one gets

$$\operatorname{Ext}^i_R(M,k) = \bigoplus_{j\in\mathbb{Z}} \operatorname{Ext}^i_R(M,k)^j = \operatorname{Hom}_k((F_i/\mathfrak{m}F_i)_j, k) \quad \text{for each} \quad i \geq 0\,.$$

Composition products turn $\mathcal{E} = \bigoplus_{i\geqslant 0, j\geqslant 0} \operatorname{Ext}^i_R(k,k)^j$ into a bigraded $k$-algebra, and $\mathcal{M} = \bigoplus_{i\geqslant 0, j\in\mathbb{Z}} \operatorname{Ext}^i_R(M,k)^j$ into a bigraded $\mathcal{E}$-module.

The $(i,j)$th *graded Betti number* of $M$ is defined to be

$$\beta^R_{i,j}(M) = \operatorname{rank}_k \operatorname{Ext}^i_R(M,k)^j\,.$$

The *graded Poincaré series* of $M$ over $R$ is the formal power series

$$P^R_M(s,t) = \sum_{i\in\mathbb{N}, j\in\mathbb{Z}} \beta^R_{i,j}(M)\, s^j t^i \in \mathbb{Z}[s^{\pm 1}][\![t]\!]\,.$$

We also use non-graded versions of these notions, namely

$$\beta^R_i(M) = \sum_{j\in\mathbb{Z}} \beta^R_{i,j}(M) \quad \text{and} \quad P^R_M(t) = \sum_{i\in\mathbb{N}} \beta^R_i(M) t^i = P^R_M(1,t) \in \mathbb{Z}[\![t]\!]\,.$$

## 1. Short linear modules and Koszul algebras

In this section $(R, \mathfrak{m}, k)$ is a graded algebra. We recall the definitions of the algebras and modules of principal interest for this paper; see [9] or [19] for details.

**1.1.** We say that an $R$-module $M$ is *linear* if it is graded and $\beta^R_{i,j}(M) = 0$ holds for all $j - i \neq d$ and some $d \in \mathbb{Z}$; in case $M \neq 0$ one has $d = \inf M$, and $M$ is generated in degree $d$. It is well-known that $M$ is linear if and only it satisfies

$$P^R_M(s,t) \cdot H_R(-st) = (-t)^{-d} H_M(-st)\,. \tag{1.1.1}$$

A linear module $M$ with $\inf M = 0$ is also called a *Koszul module.*

**1.2.** The algebra $R$ is *Koszul* if $k$ is a linear $R$-module; the equalities $\beta^R_{0,j}(\mathfrak{m}) = \beta^R_{1,j}(k)$ show that then $R$ is *standard*; that is, it is generated over $k$ by elements of degree 1. It is well-known, see [3, 1.16], that $R$ is Koszul if and only if it satisfies

$$P^R_k(t) \cdot H_R(-t) = 1\,, \tag{1.2.1}$$

if and only if the $k$-algebra $\mathcal{E}$ is generated by $\mathcal{E}^{1,1}$; see [18, Thm. 1.2] or [19, Ch. 2, §1].

We frequently refer to the following criterion:

**1.3.** If $Q$ is a standard graded $k$-algebra and $g$ is a non-zero-divisor in $Q_1$ or $Q_2$, then $Q$ and $Q/(g)$ are Koszul simultaneously, see [3, Thm. 4(e)(iv)] or [19, 6.3].

To link linearity of $M$ to linearity of $k$, we recall a construction.

**1.4.** Let $M \neq 0$ be a graded $R$-module, and set $d = \inf M$. The *trivial extension* $R \ltimes M$ has $R \oplus M(d-1)$ as graded $k$-spaces, and

$$(r_1, m_1) \cdot (r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1) \quad \text{for all } r_j \in R \text{ and } m_j \in M(d-1)\,.$$

Setting $\mathfrak{m} \ltimes M = \mathfrak{m} \oplus M(d-1)$, we get a graded $k$-algebra $(R \ltimes M, \mathfrak{m} \ltimes M, k)$.

One has $R \ltimes M = R \ltimes (M(n))$ for every $n \in \mathbb{Z}$, and the following equality holds:

$$H_{R\ltimes M}(s) = H_R(s) + s^{1-d} H_M(s)\,. \tag{1.4.1}$$

The graded version of a result of Gulliksen, see [14, Thm. 2], reads

$$P_k^{R\ltimes M}(s,t) = \frac{P_k^R(s,t)}{1 - s^{1-d} t P_M^R(s,t)} \tag{1.4.2}$$

The implication (i) $\implies$ (ii) in the next proposition is obtained in [19, Ch. 2, 5.5] by a different argument, which works also in a non-commutative situation.

**Proposition 1.5.** *Let $(R, \mathfrak{m}, k)$ be a graded algebra and $M$ a graded $R$-module. The following statements then are equivalent:*

(i) *$R$ is a Koszul algebra and $M$ is linear.*
(ii) *$R \ltimes M$ is a Koszul algebra.*
(iii) *$R$ is a Koszul algebra, and for some $d \in \mathbb{Z}$ one has*

$$P_M^R(t) \cdot H_R(-t) = (-t)^{-d} H_M(-t)\,.$$

*Proof.* (i) $\implies$ (iii). The desired equality is obtained from (1.1.1) by setting $s = 1$.

(iii) $\implies$ (ii). Comparing the orders of the formal Laurent series in (iii), one gets $d = \inf M$. In the following string of equalities, the first one comes from setting $s = 1$ in (1.4.2), the second from the hypothesis, the last one from (1.4.1).

$$\begin{aligned}
P_k^{R\ltimes M}(t) &= P_k^R(t) \cdot \frac{1}{1 - t P_M^R(t)} \\
&= \frac{1}{H_R(-t)} \cdot \frac{1}{1 - t \cdot (-t)^{-d} H_M(-t) H_R(-t)^{-1}} \\
&= \frac{1}{H_R(-t) + (-t)^{1-d} H_M(-t)} \\
&= \frac{1}{H_{R\ltimes M}(-t)}\,.
\end{aligned}$$

By (1.2.1), the composite equality implies that $R \ltimes M$ is Koszul.

(ii) $\implies$ (i). The evident homomorphisms $R \to R \ltimes M \to R$ of graded algebras compose to the identity. As $\operatorname{Ext}^i_?(k,k)^j$ is a functor of the ring argument, $\operatorname{Ext}^i_R(k,k)^j$ is a direct summand of $\operatorname{Ext}^i_{R\ltimes M}(k,k)^j$. Thus, $R$ is Koszul, so both $P_k^R(s,t)$ and $P_k^{R\ltimes M}(s,t)$ can be written as formal power series in $st$. The equality

$$P_M^R(s,t) = s^d \cdot \frac{1}{st}\left(1 - \frac{P_k^R(s,t)}{P_k^{R\ltimes M}(s,t)}\right),$$

which comes from (1.4.2), gives $\beta^R_{i,j}(M) = 0$ for $j - i \neq d$; thus, $M$ is linear. $\square$

A graded $R$-module $M$ is *short* if $H_M(s) = (p + qs)s^d$ for some $d \in \mathbb{Z}$. Koszul algebras have short linear modules, for $k$ is one, by definition. Conversely:

**Theorem 1.6.** *Let $R$ be a standard graded algebra.*

*If $R$ has a linear module $M \neq 0$ that is short, then $R$ is Koszul.*

*Proof.* Let $\varepsilon\colon R/\mathfrak{m}^2 \to k$ denote the canonical surjection. Roos [20, Cor. 1, p. 291] proves that $\operatorname{Ker}(\operatorname{Ext}^*_R(\varepsilon, k))$ is the subalgebra of $\mathcal{E} = \operatorname{Ext}^*_R(k, k)$ generated by $\operatorname{Ext}^1_R(k, k)$, so it suffices to prove that $\operatorname{Ext}^i_R(\varepsilon, k)^j = 0$ holds for all $i, j$; see 1.2.

Replacing $M$ with $M(d)$, we may assume that $M$ is Koszul. In an exact sequence

$$0 \to N \to R^b \to M \to 0$$

of graded modules with $\mathfrak{m}N \subseteq R^b$, one has $\mathfrak{m}N = N_{\geqslant 2} = R^n_{\geqslant 2} = \mathfrak{m}^2 R^n$ because $N_1$ generates $N$ and $M_2 = 0$. Thus, we get a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}N \arrow[r] \arrow[d] & R^b \arrow[r] \arrow[d, equal] & (R/\mathfrak{m}^2)^b \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] & R^b \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

It induces the square in the following commutative diagram, where the factorization $(R/\mathfrak{m}^2)^b \to M \to k^b$ of $\varepsilon^b\colon (R/\mathfrak{m}^2)^b \to k^b$ induces the triangle:

$$\begin{tikzcd}
 & \operatorname{Ext}^i_R((R/\mathfrak{m}^2)^b, k)^j & \operatorname{Ext}^{i-1}_R(\mathfrak{m}N, k)^j \arrow[l, "\cong"'] \\
\operatorname{Ext}^i_R(k^b, k)^j \arrow[ur, bend left, "\operatorname{Ext}^i_R(\varepsilon^b, k)^j"] \arrow[dr, bend right] & & \\
 & \operatorname{Ext}^i_R(M, k)^j \arrow[uu] & \operatorname{Ext}^{i-1}_R(N, k)^j \arrow[l, "\cong"'] \arrow[uu, "0"']
\end{tikzcd}$$

One has $\operatorname{Ext}^{i-1}_R(N, k)^j = 0$ for $j \neq i$ by isomorphism in the bottom row, and $\operatorname{Ext}^{i-1}_R(\mathfrak{m}N, k)^i = 0$ because $\inf(\mathfrak{m}N) = 2$, so the vertical map on the right is zero. Now the diagram implies $\operatorname{Ext}^i_R(\varepsilon^b, k) = 0$, whence $\operatorname{Ext}^i_R(\varepsilon, k) = 0$. $\square$

We say that an algebra $R$ is *short* when $R_i = 0$ for $i \geq 3$.

Existence of linear modules imposes numerical constraints on short algebras.

**Corollary 1.7.** *Let $R$ be a standard graded with $H_R(s) = 1 + es + rs^2$, and set*

$$u = (e - \sqrt{e^2 - 4r})/2 \quad \textit{and} \quad v = (e + \sqrt{e^2 - 4r})/2\,.$$

*If there exists a non-free linear $R$-module, then $R$ is Koszul and $e^2 \geq 4r$ holds.*

*If $M$ is a linear $R$-module with $H_M(s) = ps^d + qs^{d+1}$ and $p \neq 0$, then $q \leq vp$.*

*If, furthermore, $q = vp$, then $u$ and $v$ are integers, and there is an equality*

$$P^R_M(s, t) = \frac{ps^d}{1 - ust}\,.$$

*Proof.* Let $N$ be a non-free linear $R$-module, and set $j = \inf(N)$. One then has $0 \neq \Omega^R_1(N) \subseteq \mathfrak{m}R^n(-j)$, so $N$ is short, non-zero, and linear. Theorem 1.6 shows that $R$ is Koszul, and then $e^2 \geq 4r$ holds by Conca, Trung, and Valla [7, 3.4].

When $H_M(s) = ps^d + qs^{d+1}$, from formula (1.1.1) we get

$$P^R_M(s,t) = \frac{s^d(p-qst)}{1-est+rst^2} = \frac{s^d(p-qst)}{(1-ust)(1-vst)}\,.$$

Prime fraction decomposition yields real numbers $a$ and $b$, such that

$$P^R_M(s,t) = \begin{cases} s^d\sum_{i=0}^{\infty}(au^i+bv^i)(st)^i & \text{when } e^2 > 4r\,, \\ s^d\sum_{i=0}^{\infty}(av^i+b(i+1)v^i)(st)^i & \text{when } e^2 = 4r\,. \end{cases}$$

As $M$ is not free, the coefficient of $s^d(st)^i$ is positive for each $i \geq 0$, hence $b \geq 0$.

When $e^2 > 4r$ one has $a+b = p$ and $av+bu = q$, and hence $pv = q+b(v-u) \geq q$. When $e^2 = 4r$ holds, $a+b = p$ and $av = q$ give $pv = q+bv \geq q$.

Assume $pv = q$. In both cases one then has $b = 0$, which means $P^R_M(s,t) = ps^d(1-ust)^{-1}$. This implies that $u$ is an integer, and hence so is $v = e-u$. □

We illustrate the tightness of the hypotheses in the last two results:

**Example 1.8.** Let $k$ be a field and set $R = k[x,y]/(x^2,y^3)$.

The algebra $R$ satisfies $R_i = 0$ for $i \geq 4$, and the $R$-module $M = R/xR$ is non-free, linear, with $M_n = 0$ for $n \neq 0,1,2$. However, $R$ is not Koszul.

*Remark* 1.9. If $V$ is a *generic* $k$-subspace of codimension $r$ in the space of quadrics in $k[x_1,\dots,x_e]$. By [7, 3.1], $e^2 \geq 4r$ implies that $k[x_1,\dots,x_e]/(V)$ is short and Koszul; the converse also holds, due to Fröberg and Löfwall [10, 7.1].

Partial versions of the Koszul property are also of interest.

We say that an $R$-module $M$ is *$m$-step linear* for some integer $m \geq 0$ if it satisfies $\beta^R_{i,j}(M) = 0$ for all $j \neq i + \inf M$ with $i \leq m$. Thus, 0-step linear means that $M$ is generated in a single degree, and 1-step linear means that, in addition, it has a free presentation with a presenting matrix of linear forms.

**Proposition 1.10.** *Let $R$ be a Koszul algebra and $M$ an $R$-module with* $\inf M = 0$.

*For every non-negative integer $m$ the following conditions are equivalent.*

(i) *$M$ is $m$-step linear.*

(ii) $P^R_M(s,t) \equiv H_M(-st)\cdot H_R(-st)^{-1} \pmod{t^{m+1}}$

*When $R$ is short, they are also equivalent to*

(iii) $\beta^R_{m,m+1}(M) = 0$

*Proof.* (i) $\Longrightarrow$ (ii). By hypothesis, there is an exact sequence

$$0 \to L \to R(-m)^{b_m} \to \cdots \to R(-1)^{b_1} \to R^{b_0} \to M \to 0$$

of $R$-modules with $b_i = \beta^R_{i,i}(M)$ for $i \leq m$, and $L_j = 0$ for $j \leq m$. It yields

$$H_M(s) = \sum_{i=0}^{m}(-1)^i b_i s^i H_R(s) + (-1)^{m+1}H_L(s)\,.$$

Dividing this equality by $H_R(s)$ and replacing $s$ with $-st$, one gets (ii).

(ii) $\Longrightarrow$ (i) $\Longrightarrow$ (iii). These implications hold by definition.

(iii) $\Longrightarrow$ (ii). Let $\mathcal{E}$ be the bigraded algebra $\operatorname{Ext}^*_R(k,k)$ and $\mathcal{M}$ its bigraded module $\operatorname{Ext}^*_R(M,k)$. Write $(M_0)^*$ for the bigraded $k$-vector space with $\operatorname{Hom}_k(M_0,k)$ in bidegree $(0,0)$ and 0 elsewhere, and $(M_1)^*$ for that with $\operatorname{Hom}_k(M_1,k)$ in bidegree $(1,1)$ and 0 elsewhere. Graded versions of [2, 2.4, 2.5] yield an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}\otimes_k (M_1)^* \longrightarrow (\mathcal{E}\otimes_k (M_0)^*) \oplus \Sigma\mathcal{F} \longrightarrow \mathcal{M} \longrightarrow 0$$

of bigraded $\mathcal{E}$-modules, where $\mathcal{F}$ is free, and $(\Sigma\mathcal{F})^{i,j} = \mathcal{F}^{i+1,j}$.

One has $\mathcal{E}^{i,j} = 0$ for $i \neq j$, because $R$ is Koszul, and $\mathcal{F}^{i,j} = 0$ for $i \neq j$, because of the inclusion $\mathcal{F} \subseteq \mathcal{E} \otimes_k (M_1)^*$. Set $r_l = \operatorname{rank}_k \mathcal{F}^{l,l}$. The exact sequence gives

$$P^R_M(s,t) = H_M(-st) \cdot P^R_k(s,t) + \left(1 + \frac{1}{t}\right) \cdot \sum_{l=0}^{\infty} r_l\, (st)^l \,.$$

Since $R$ is Koszul, (1.1.1) gives $P^R_k(s,t) = H_R(-st)^{-1}$, so the formula above yields

$$P^R_M(s,t) - H_M(-st) \cdot H_R(-st)^{-1} = \sum_{l=0}^{\infty} (r_l + s r_{l+1})\, (st)^l \,.$$

As $\mathcal{E}^i \neq 0$ for $i \geq 0$ and $\mathcal{F}$ is a free $\mathcal{E}$-module, $r_l = 0$ means $r_i = 0$ for $i \leq l$. $\square$

## 2. Parametrizing short modules

In this section $R$ is a standard graded algebra with $R_1 \neq 0$ and $x_1, \dots, x_e$ a fixed $k$-basis of $R_1$. Let $p$ be a positive integer and $q$ a non-negative one. The goal here is to describe a convenient parameter space for modules with Hilbert series $p + qs$.

**2.1.** Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_h\}_{h\in\mathbb{N}}$ be the standard bases of the vector spaces $k^{(\mathbb{N})}$ and $k^{(\mathbb{N})}(-1)$, respectively. For each pair $(p,q)$ of non-negative integers, let $k^{p,q}$ denote the $k$-linear span in $k^{(\mathbb{N})} \oplus (k^{(\mathbb{N})}(-1))$ of $\{u_1, \dots, u_p\} \cup \{v_1, \dots, v_q\}$.

Note that one has $k^{p,q} \subseteq k^{p',q'}$ for $p \leq p'$ and $q \leq q'$.

**2.2.** Let $[1,p]$ denote the set $\{1, \dots, p\}$ of natural numbers with the natural order, and order the elements of $[1,e] \times [1,p]$ lexicographically:

$$(l,n) < (l',n') \iff l < l' \text{ or } (l = l' \text{ and } n < n')\,. \tag{2.2.1}$$

For $q \geq 1$ we let $\mathrm{M}_{ep\times q}(k)$ denote the set of $ep \times q$ matrices with entries in $k$, with rows indexed by the elements of $[1,e] \times [1,p]$ and columns by those of $[1,q]$. *We identify* $\mathrm{M}_{ep\times q}(k)$ *with the affine space* $\mathbb{A}^{epq}_k$ *over* $k$*, endowed with the Zariski topology*; by convention, we extend this identification to the case $q = 0$.

For every subset $\boldsymbol{s} \subseteq [1,e] \times [1,p]$ and $C \in \mathrm{M}_{ep\times q}(k)$, let $C_{\boldsymbol{s}}$ denote the $|\boldsymbol{s}| \times q$ submatrix of $C$ with rows indexed by $\boldsymbol{s}$; thus $C_{(l,n)}$ is the $(l,n)$th row of $C$.

When $|\boldsymbol{s}| = q$, we form the following collection of matrices:

$$\mathrm{M}_{ep\times q}(k)(\boldsymbol{s}) = \{C \in \mathrm{M}_{ep\times q}(k) \mid \det(C_{\boldsymbol{s}}) \neq 0\}\,. \tag{2.2.2}$$

This is a basic open subset of $\mathrm{M}_{ep\times q}(k)$, which is empty when $q > ep$. The subset

$$\boldsymbol{L}^0_{p,q} = \{C \in \mathrm{M}_{ep\times q}(k) \mid \operatorname{rank} C = q\} \tag{2.2.3}$$

is open $\mathrm{M}_{ep\times q}(k)$, and is covered by the basic open subsets above:

$$\boldsymbol{L}^0_{p,q} = \bigcup_{\substack{\boldsymbol{s}\subseteq[1,e]\times[1,p] \\ |\boldsymbol{s}|=q}} \mathrm{M}_{ep\times q}(k)(\boldsymbol{s})\,. \tag{2.2.4}$$

We parametrize short $R$-modules by means of their *multiplication tables*.

**2.3.** To each $R$-module $M$ with underlying vector space $k^{p,q}$ we associate the matrix $C^R = (c_{(l,n),h})$ in $\mathrm{M}_{ep\times q}(k)$, with $(l,n)$th row defined by the equality

$$x_l u_n = \sum_{h=1}^{q} c_{(l,n),h} v_h \quad \text{for each} \quad (l,n) \in [1,e] \times [1,p]\,.$$

Conversely, each matrix $C = (c_{(l,n),h}) \in \mathrm{M}_{ep\times q}(k)$ defines, through the formula above, an action of $R$ on $k^{p,q}$ that turns it into an $R$-module, called $R^C$.

The maps described above clearly are mutually inverse.

The correspondence in 2.3 allows one to shuttle between $R$-module structures on $k^{p,q}$ and $ep \times q$ matrices with elements in $k$. In particular, we identify $\boldsymbol{L}^0_{p,q}$ with the set of 0-step linear module structures on $k^{p,q}$.

Graded $R$-modules are often parametrized in terms of their *minimal presentations* over $R$. This format is not used below, but we pause to show that results on non-empty open loci faithfully translate between parametrizations.

**2.4.** For every matrix $B = (b_{(l,n),h'})$ in $\mathrm{M}_{ep\times(ep-q)}(k)$, let $\varkappa^B_1\colon k^{ep-q} \to R_1 \otimes_k k^p$ denote the homomorphism of $k$-vector spaces given by the formula

$$w_{h'} \mapsto \sum_{(l,n)\in[1,e]\times[1,p]} b_{(l,n),h'}\, x_l \otimes u_n \quad \text{for} \quad h' = 1,\dots,ep-q\,,$$

where $w_1,\dots,w_{ep-q}$ denote the standard basis of $k^{ep-q}$.

For every matrix $C = (c_{(l,n),h})$ in $\mathrm{M}_{ep\times q}(k)$, let $\lambda^C_1\colon R_1 \otimes_k k^p \to k^q$ denote the homomorphism of $k$-vector spaces given by the formula

$$x_l \otimes u_n \mapsto \sum_{h=1}^{q} c_{(l,n),h} v_h \quad \text{for} \quad (l,n) \in [1,e]\times[1,p]\,.$$

Define an open subset of $\mathrm{M}_{ep\times(ep-q)}(k)$ by setting

$$\boldsymbol{K}^0_{p,q} = \{B \in \mathrm{M}_{ep\times(ep-q)}(k) \mid \mathrm{rank}_k(B) = ep-q\}\,.$$

The assignments $B \mapsto \mathrm{Im}(\varkappa(B))$ and $C \mapsto \mathrm{Ker}(\lambda(C))$ define morphisms of algebraic varieties to the Grassmannian of $(ep-q)$-dimensional subspaces of $R_1 \otimes_k k^p$:

$$\varkappa\colon \boldsymbol{K}^0_{p,q} \longrightarrow \mathrm{Grass}_{(ep-q)}(R_1 \otimes_k k^p) \longleftarrow \boldsymbol{L}^0_{p,q} : \lambda$$

By construction, these maps above are morphisms of algebraic varieties.

An important point here is that $\varkappa$ and $\lambda$ are *open*. This follows from a classical theorem of Chevalley, because both maps are dominant (being surjective), the closed fibers of each one have constant dimension (being isomorphic to some fixed affine space), and Grassmann varieties are normal (being smooth). Modern proofs of Chevalley's Theorem are not easy to find. Instead, we refer to a much more general statement proved by Grothendieck in [13, 14.4.4(c)], which contains the one used here; see [12, 6.15.1]. We thank Joseph Lipman for help with these references.

Every matrix $B = (b_{(l,n),h'})$ in $\mathrm{M}_{ep\times(ep-q)}(k)$ yields a homomorphism of graded $R$-modules $\varkappa^B\colon R \otimes_k k^{ep-q}(-1) \to R \otimes_k k^p$, equal to $\varkappa^B_1$ in degree 1. The subsets

$$\boldsymbol{K}^1_{p,q}(R) = \{B \in \mathrm{M}_{ep\times(ep-q)}(k) \mid \varkappa^B_1 \text{ is injective and } \varkappa^B_2 \text{ is surjective}\}$$
$$\boldsymbol{L}^1_{p,q}(R) = \{C \in \mathrm{M}_{ep\times q}(k) \mid \ \lambda^C_1 \text{ is injective}\}\,,$$

of $\mathrm{M}_{ep\times(ep-q)}(k)$ are open. One has $H_{\mathrm{Coker}(\varkappa^B)}(s) = p+qs$, so $\boldsymbol{K}^1_{p,q}(R)$ and $\boldsymbol{L}^1_{p,q}(R)$ parametrize the same set of $R$-modules. The parametrizations are interchangeable:

**Lemma 2.5.** *When $k$ is an algebraically closed field and $R$ a standard graded $k$-algebra the maps in 2.4 restrict to open morphisms of algebraic varieties*

$$\varkappa\colon \boldsymbol{K}^1_{p,q}(R) \longrightarrow \mathrm{Grass}_{(ep-q)}(R_1 \otimes_k k^p) \longleftarrow \boldsymbol{L}^1_{p,q}(R) : \lambda$$

*with the same image, so $U \subseteq \boldsymbol{K}^1_{p,q}(R)$ (respectively, $U \subseteq \boldsymbol{L}^1_{p,q}(R)$) is open or non-empty if and only if $\lambda^{-1}\varkappa(U) \subseteq \boldsymbol{L}^1_{p,q}(R)$ (respectively, $\varkappa^{-1}\lambda(U) \subseteq \boldsymbol{K}^1_{p,q}(R)$) is.*

*Proof.* Only the assertion concerning the images needs validation. For each matrix $C$ in $\mathrm{M}_{ep\times q}(k)$, let $\lambda^C \colon R(-1)\otimes_k \mathrm{Ker}(\lambda(C)) \to R\otimes_k k^p$ be the $R$-linear map, equal to $\lambda^C_1$ in degree 1. One has $C \in \boldsymbol{L}^1_{p,q}(R)$ if and only if $\lambda^C_1$ is injective and $\lambda^C_2$ is surjective. Comparison of definitions gives $\lambda(\boldsymbol{L}^1_{p,q}(R)) = \varkappa(\boldsymbol{K}^1_{p,q}(R))$. □

## 3. Linear loci

Let $R$ be a standard graded $k$-algebra, and set

$$e = \operatorname{rank}_k R_1 \quad \text{and} \quad r = \operatorname{rank}_k R_2 \,.$$

**3.1.** Fix positive integers $p$ and $q$. The *linear locus* of $R$ in $\mathrm{M}_{ep\times q}(k)$ is the subset

$$\boldsymbol{L}_{p,q}(R) = \{C \in \mathrm{M}_{ep\times q}(k) \mid \text{the } R\text{-module } R^C \text{ is Koszul}\}\,,$$

where $R^C$ is the graded $R$-module defined in 2.3.

Our goal is to identify conditions for $\boldsymbol{L}_{p,q}(R)$ to have a *non-empty interior*; that is, for it to contain a non-empty open subset.

**Theorem 3.2.** *Let $R$ be a Koszul algebra and $p' \in [1,p]$ and $q' \in [q,ep]$ be integers. If $\boldsymbol{L}_{p',q}(R)$ or $\boldsymbol{L}_{p,q'}(R)$ has a non-empty interior, then so does $\boldsymbol{L}_{p,q}(R)$.*

In the proof we use the functoriality of the correspondence in 2.3:

**3.3.** For $p' \in [1,p]$, let $\iota \colon k^{p',q} \hookrightarrow k^{p,q}$ denote the inclusion map.

As $k^{p',q}$ is a submodule for every $R$-module structure on $k^{p,q}$, we get a map

$$\iota^* \colon \mathrm{M}_{ep\times q}(k) \to \mathrm{M}_{ep'\times q}(k) \tag{3.3.1}$$

of affine spaces over $k$, which is linear and surjective: It sends each $ep\times q$ matrix to the $(ep')\times q$ submatrix with rows indexed by the pairs $(l,n)$ with $n\in[1,p']$.

For $q' \in [q,ep]$, let $\pi \colon k^{p,q'} \twoheadrightarrow k^{p,q}$ be the projection with $\pi(v_h)=0$ for $h \geq q+1$.

Since $\mathrm{Ker}(\pi)$ is a submodule for every $R$-module structure on $k^{p,q'}$, we get a map

$$\pi_* \colon \mathrm{M}_{ep\times q'}(k) \to \mathrm{M}_{ep\times q}(k) \tag{3.3.2}$$

of affine spaces over $k$, which is linear and surjective: It sends every $ep\times q'$ matrix to its $ep\times q$ submatrix, whose columns are indexed by the elements in $[1,q]$.

**Lemma 3.4.** *When $R$ is a Koszul algebra the maps (3.3.1) and (3.3.2) satisfy*

$$\pi_*(\boldsymbol{L}_{p,q'}(R)) \subseteq \boldsymbol{L}_{p,q}(R) \supseteq (\iota^*)^{-1}(\boldsymbol{L}_{p',q}(R))$$

*Proof.* For $C \in \mathrm{M}_{ep\times q}(k)$, the construction in 3.3 gives an exact sequence

$$0 \to R^{\iota^*(C)} \to R^C \to N \to 0$$

where $N_j = 0$ for $j\neq 0$. If $R^{\iota^*(C)}$ Koszul, then in the induced exact sequence

$$\mathrm{Ext}^i_R(N,k)^j \to \mathrm{Ext}^i_R(R^C,k)^j \to \mathrm{Ext}^i_R(R^{\iota^*(C)},k)^j$$

both extremal terms are zero for $j\neq i$, because $R$ is Koszul.

For $C' \in \mathrm{M}_{ep\times q'}(k)$, the construction in 3.3 gives an exact sequence

$$0 \to L \to R^{C'} \to R^{\pi_*(C')} \to 0$$

where $L_j = 0$ for $j \neq 1$. When $R^{C'}$ is Koszul, in the induced exact sequence

$$\mathrm{Ext}_R^{i-1}(L,k)^j \to \mathrm{Ext}_R^i(R^{\pi_*(C')},k)^j \to \mathrm{Ext}_R^i(R^{C'},k)^j$$

both extremal terms are zero for $j \neq i$, because $R$ is Koszul. □

*Proof of Theorem 3.2.* Let $U \subseteq \boldsymbol{L}_{p',q}(R)$ be non-empty and open in $\mathrm{M}_{ep'\times q}(k)$. The subset $(\iota^*)^{-1}(U)$ of $\mathrm{M}_{ep\times q}(k)$ is open, because $\iota^*$ is continuous, and not empty, because $\iota^*$ is surjective. By Lemma 3.4, it is contained in $\boldsymbol{L}_{p,q}(R)$.

Let $\sigma\colon \mathrm{M}_{ep\times q}(k) \to \mathrm{M}_{ep\times q'}(k)$ be the map that sends every $C \in \mathrm{M}_{ep\times q}(k)$ to the $ep \times q'$ matrix, obtained by the addition of zero columns with indices $q+1,\dots,q'$. Let $U' \subseteq \boldsymbol{L}_{p,q'}(R)$ be non-empty and open in $\mathrm{M}_{ep\times q'}(k)$, and pick $C' \in U'$. In the affine subspace $W = C' + \sigma(\mathrm{M}_{ep\times q}(k))$ of $\mathrm{M}_{ep\times q'}(k)$, the set $U' \cap W$ is non-empty and open. Since $\pi_*|_W\colon W \to \mathrm{M}_{ep\times q}(k)$ is a homeomorphism, $\pi_*(U'\cap W)$ is non-empty and open in $\mathrm{M}_{ep\times q}(k)$. By Lemma 3.4, it is contained in $\boldsymbol{L}_{p,q}(R)$. □

Within a given parameter space $\mathrm{M}_{ep\times q}(k)$, it is sometimes possible to transfer information between linear loci of different $k$-algebras. We give an example.

**Proposition 3.5.** *Let $\rho\colon R' \to R$ be a homomorphism of graded $k$-algebras.*
*If $\rho$ is a Golod homomorphism and $R'$ is Koszul, then for all $p$ and $q$ one has*

$$\boldsymbol{L}_{p,q}(R) \supseteq \boldsymbol{L}_{p,q}(R')\,.$$

*Proof.* This follows from [2, 3.3], where it is proved that if an $R$-module $M$ is Koszul when considered as a module over $R'$ via $\rho$, then $M$ is Koszul over $R$. □

**3.6.** We approximate the linear locus of $R$, *from above*, by the sets

$$\boldsymbol{L}^m_{p,q}(R) = \{C \in \mathrm{M}_{ep\times q}(k) \mid R^C \text{ is } m\text{-step linear}\}\,,$$

defined for every integer $m \geq 0$. The following inclusions are evident:

$$\boldsymbol{L}^m_{p,q}(R) \supseteq \boldsymbol{L}^{m+1}_{p,q}(R) \quad\text{and}\quad \boldsymbol{L}_{p,q}(R) = \bigcap_{m\geqslant 0} \boldsymbol{L}^m_{p,q}(R)\,. \tag{3.6.1}$$

For completeness, we include the proof of a folklore result; stronger ones have been communicated to us by David Eisenbud, Anthony Iarrobino, and Clas Löfwall.

**Lemma 3.7.** *When $R$ is Koszul $\boldsymbol{L}^m_{p,q}(R)$ is open in $\mathrm{M}_{ep\times q}(k)$ for every $m \geq 0$.*

*Proof.* Pick a matrix $C$ in $\mathrm{M}_{ep\times q}(k)$.

One has $\beta^R_{i,j}(R^C) = \mathrm{rank}_k\,\mathrm{Tor}^R_i(R^C,k)_j$, so we fix $m \geq 0$ and prove that the subset of $\mathrm{M}_{ep\times q}(k)$, defined by $\mathrm{Tor}^R_i(R^C,k)_j = 0$ for $j \neq i \leq m$, is open.

Let $G$ be a minimal free resolution of $k$ over $R$. As $R$ is Koszul, we may assume $G_i = R(-i)^{b_i}$ with $b_i = \beta^R_i(k)$. As $R^C$ is short, one has $(R^C \otimes_R G_i)_j = 0$ for $j \neq i, i+1$. This yields $\mathrm{Tor}^R_i(R^C,k)_j = 0$ for $j \neq i, i+1$, and an exact sequence

$$0 \to \mathrm{Tor}^R_i(R^C,k)_i \to (R^C_0)^{b_i} \xrightarrow{\delta_i} (R^C_1)^{b_{i-1}} \to \mathrm{Tor}^R_{i-1}(R^C,k)_i \to 0$$

of $k$-vector spaces for every $i \geq 0$, where $\delta_i = (R^C_0 \otimes_R \partial^G_i)_i$.

For each $i$, let $G^{(i)}$ denote the standard basis of $R(-i)^{b_i}$ over $R$. In these bases $\partial^G_i$ is given by a matrix of linear forms in $x_1,\dots,x_e$. In the $k$-bases

$$\{u_n \otimes g^{(i)} \mid n = 1,\dots,p\,;\ g^{(i)} \in G^{(i)}\} \quad\text{and}$$
$$\{v_h \otimes g^{(i-1)} \mid h = 1,\dots,q\,;\ g^{(i-1)} \in G^{(i-1)}\}$$

of $(R_0^C)^{b_i}$ of $(R_1^C)^{b_{i-1}}$, respectively, the map $\delta_i$ then is given by a matrix of linear forms in the elements $c_{(l,n),j}$ from the multiplication table in 2.3. The condition $\operatorname{Tor}_i^R(R^C,k)_{i+1}=0$ for $i\le m$ is equivalent to the surjectivity of $\delta_i$ for $i\le m+1$. The latter condition means that some maximal minor of the matrices $\delta_1,\dots,\delta_{m+1}$ is different from 0, and so determines an open subset of $\mathrm{M}_{ep\times q}(k)$. □

## 4. Periodic linear modules

In this section $R$ denotes a standard graded $k$-algebra. We say that a graded $R$-module $M$ is *linear of period* 2 if there is an exact sequence of graded $R$-modules

$$0\to M(-2)\to R^p(-d-1)\to R^p(-d)\to M\to 0 \tag{4.0.1}$$

Splicing suitable shifts of the exact sequence above, one sees that $M$ is linear.

We explore the interplay of periodicity, linearity, and shortness. When $N$ is an $R$-module, $\Omega_i^R(N)$ denotes the $i$th module of syzygies in a minimal free resolution of $N$. Assuming $R_3=0$, Lescot [17, 3.4] established part (1) of the next theorem.

**Theorem 4.1.** *Let $R$ be standard graded $k$-algebra with $\operatorname{rank}_k R_1=e\ge 1$, let $M$ be a non-zero $R$-module with $\inf M=d$, and $p$ a positive integer.*

(1) *The following conditions are equivalent.*
  (i) $P_M^R(s,t)=ps^d\cdot(1-st)^{-1}$ *and $M$ is short.*
  (ii) *$M$ is linear over $R$ with $H_M(s)=(1+(e-1)s)\cdot ps^d$, and one has $H_R(s)=(1+(e-1)s)\cdot(1+s)$.*
  *They imply that $R$ is Koszul.*

(2) *If $Q$ is a standard graded $k$-algebra with $\operatorname{rank}_k Q_1=e$ and $\psi\colon Q\to R$ is a surjective homomorphism of algebras, then the following conditions are equivalent.*
  (iii) $P_M^Q(s,t)=ps^d\cdot(1-(st)^{c+1})\cdot(1-st)^{-1}$ *for some $c\ge 1$, and $M$ is short.*
  (iv) *$M$ is linear over $Q$ with $H_M(s)=(1+(e-1)s)\cdot ps^d$, and one has $H_Q(s)=(1+(e-1)s)\cdot(1-s)^{-1}$.*
  (v) $P_M^Q(s,t)=ps^d\cdot(1+st)$ *and* $H_Q(s)=(1+(e-1)s)\cdot(1-s)^{-1}$.

  *They imply that $Q$ is Koszul, Golod, and Cohen-Macaulay of dimension* 1.

(3) *If $g\in Q_2$ is a non-zero-divisor and $\operatorname{Ker}(\psi)=(g)$ in* (2)*, then the condition*

  (vi) *$M$ is linear of period 2 over $R$ and $H_R(s)=(1+(e-1)s)\cdot(1+s)$.*

  *satisfies the implications* (v) $\Longrightarrow$ (vi) $\Longleftrightarrow$ (i)*, and also* (v) $\Longleftrightarrow$ (vi) *if $e\ne 2$.*

*Proof.* When $M$ is short we write $H_M(s)$ in the form $s^d(a+bs)$.

(1) (i) $\Longrightarrow$ (ii). The algebra $R$ is Koszul by Theorem 1.6, so using (1.1.1) we get

$$pH_R(st)=\frac{pH_M(st)}{t^d\cdot P_M^R(s,-t)}=(a+bst)(1+st)\,.$$

The expressions for $H_M(s)$ and $H_R(s)$ follow by comparing degrees and coefficients.

(ii) $\Longrightarrow$ (i). This implication follows directly from (1.1.1).

(2) (iii) $\Longrightarrow$ (iv). The algebra $Q$ is Koszul by Theorem 1.6, so (1.1.1) gives

$$pH_Q(st)=\frac{pH_M(st)}{t^d\cdot P_M^Q(s,-t)}=\frac{(a+bst)(1+st)}{1-(-st)^{c+1}}\,.$$

Recall that $H_Q(s)$ can be written as $h(s)/(1-s)^n$ with $h(s)$ in $\mathbb{Z}[s]$ satisfying $h(1) \neq 0$, and $n = \dim Q$. Setting $t = 1$ in the formula above, we obtain

$$ph(s)(1-(-s)^{c+1}) = (a+bs)(1+s)(1-s)^n \in \mathbb{Z}[s]\,.$$

Comparing orders of vanishing at $s=1$, we get $c = 1 = n$, hence $ph(s) = a+bs$. The desired expressions for $H_M(s)$ and $H_Q(s)$ follow, along with $P^Q_M(s,t) = (1+st)\cdot ps^d$. Thus, $M$ has projective dimension 1, which entails $\operatorname{depth} Q \geq 1$. The expression for $H_M(s)$ yields $\dim Q = 1$, so $Q$ is Cohen-Macaulay; it also shows that $Q$ has multiplicity $e$; as $\operatorname{edim} Q = e$ and $\dim Q = 1$, this is the minimal possible value.

(iv) $\Longrightarrow$ (v) $\Longrightarrow$ (iii). These implications follows directly from (1.1.1).

(3) The isomorphism $R \cong Q/(g)$ with $g$ a non-zero-divisor in $Q_2$ implies

$$H_R(s) = (1-s^2)H_Q(s)\,. \tag{4.1.1}$$

(v) $\Longrightarrow$ (vi). By hypothesis, there is an exact sequence of $Q$-modules

$$0 \to Q^p(-1) \to Q^p \to M \to 0$$

The resolution $0 \to Q(-2) \xrightarrow{g} Q \to 0$ of $R$ over $Q$ yields $\operatorname{Tor}^Q_1(R,M) \cong M(-2)$, so application of $\operatorname{Tor}^Q(R,-)$ to the exact sequence above yields an exact sequence of the form (4.0.1). The expression for $H_R(s)$ comes from formula (4.1.1).

(vi) $\Longrightarrow$ (i). From the exact sequence (4.0.1) we obtain

$$H_M(s) = H_R(s)\cdot ps^d(1+s)^{-1} = (1+(e-1)s)\cdot ps^d\,.$$

It follows that $M$ admits no direct summand isomorphic to $R$, so the linear free resolution of $M$ over $R$, obtained by splicing suitable shifts of (4.1.1), is minimal.

(i) $\Longrightarrow$ (v) when $e \neq 2$. If $e = 1$, then one has $Q \cong k[x]$ and $M \cong k^p$.

Assume $e \geq 3$. From [1, 3.3.4], one gets $\beta^Q_i(M) \leq 2p$ for each $i$. The ring $Q$ is Cohen-Macaulay of dimension 1. One gets $H_Q(s) = (1+(e-1)s)(1-s)^{-1}$ from (4.1.1), so $Q$ has embedding dimension $e$ and multiplicity $e$. Thus, $Q$ has *minimal multiplicity*, and so is Golod; see [1, 5.2.8]. This implies that $M$ has finite projective dimension over $Q$; see [1, 5.5.3(5)]. As $\operatorname{depth} Q = 1$, there is an exact sequence

$$0 \to \bigoplus_{j=1}^{n} Q^{r_j}(-j) \to Q^p(-d) \to M \to 0$$

The expressions for $H_Q(s)$, noted above, and for $H_M(s)$, from (ii), then yield

$$ps^d(1+(e-1)s) = H_M(s) = \left(ps^d - \sum_{j=1}^{n} r_j s^{d+j}\right)\frac{1+(e-1)s}{1-s}\,.$$

We get $ps^d(1-s) = ps^d - \sum_{j=1}^n r_j s^j$, whence $r_j = 0$ for $j \neq d+1$ and $r_{d+1} = p$.

(i) $\Longrightarrow$ (vi) when $e = 2$. We may assume $d = 0$.

By (1) the algebra $R$ is Koszul, hence quadratic, and thus $R \cong k[x,y]/(f,g)$ with $f,g$ a regular sequence of quadrics.

The hypothesis on $P^R_M(s,t)$ give an exact sequence

$$0 \to N \to R^p(-3) \xrightarrow{\alpha} R^p(-2) \xrightarrow{\beta} R^p(-1) \xrightarrow{\gamma} R^p \to M \to 0$$

As $R$ is complete intersection, one has $N \cong \operatorname{Coker}(\alpha)(-2)$ by [8, 4.1].

Set $(-)^* = \operatorname{Hom}_R(-,R)$. Since $R$ is self-injective, we get an exact sequence

$$0 \to M^* \to R^p \xrightarrow{\gamma^*} R^p(1) \xrightarrow{\beta^*} R^p(2) \xrightarrow{\alpha^*} R^p(3) \to N^* \to 0$$

and an isomorphism $\operatorname{Ker}(\alpha^*) \cong N^*(-2)$. From these data, we obtain

$$M^* \cong \Omega^2_R(\operatorname{Ker}(\alpha^*)) \cong \Omega^2_R(N^*(-2)) \cong N^*(-4)$$

As all $R$-modules are reflexive, we get $M \cong N(4) \cong \operatorname{Coker}(\alpha)(2) \cong \operatorname{Ker}(\gamma)(2)\,.$ □

We use modules of period 2 to study the linear locus $\boldsymbol{L}_{p,(e-1)p}(R)$.

**4.2.** We approximate the linear locus of $R$, *from below*, by the set

$$\boldsymbol{P}^2_{p,q}(R) = \{C \in \boldsymbol{L}_{p,q}(R) \mid R^C \text{ is of period } 2\}\,.$$

**Corollary 4.3.** *Assume $H_R(s) = 1 + es + (e-1)s^2$ with $e \geq 1$, and $R \cong Q/(g)$ for some standard graded algebra $Q$ and non-zero-divisor $g \in Q_2$.*

*For each positive integer $p$ the set $\boldsymbol{L}^2_{p,(e-1)p}(Q)$ is open in $\mathrm{M}_{ep\times(e-1)p}(k)$ and there the following inclusions hold, with equality when $e \neq 2$:*

$$\boldsymbol{L}_{p,(e-1)p}(R) = \boldsymbol{P}^2_{p,(e-1)p}(R) \supseteq \boldsymbol{L}_{p,(e-1)p}(Q) = \boldsymbol{L}^2_{p,(e-1)p}(Q)\,.$$

*If $Q_1$ contains a non-zero-divisor (e.g., if $k$ is infinite), then $\boldsymbol{L}_{p,(e-1)p}(Q) \neq \varnothing$.*

*Proof.* Every short $Q$-module is annihilated by $g$, for degree reasons, and so is also an $R$-module. With this remark, Theorem 4.1 implies the inclusion and the equalities. The set $\boldsymbol{L}^2_{p,(e-1)p}(Q)$ is open by Lemma 3.7.

If $h \in Q_1$ is a non-zero divisor, then for $N = (Q/(h))^p$ one has an exact sequence

$$0 \to Q(-1)^p \xrightarrow{h} Q^p \to N \to 0$$

that gives $H_N(s) = (p - ps)H_Q(s) = p + (e-1)ps$; thus, $N$ is in $\boldsymbol{L}^2_{p,(e-1)p}(Q)$. □

When $e = 2$, neither the implication (v) $\implies$ (vi) in Theorem 4.1, nor the inclusion in Corollary 4.3 can be reversed in general:

**Example 4.4.** Set $Q = k[x,y]/(x^2)$ and $R = Q/(y^2)$. The $R$-module $M = R/(x)$ then is linear of period 2, as demonstrated by the exact sequence

$$0 \to M(-2) \to R(-1) \xrightarrow{x} R \to M \to 0$$

However, as a $Q$-module $M$ has infinite projective dimension and is not linear.

For cyclic modules, we have an additional criterion for openness.

**Proposition 4.5.** *If $R$ is a Koszul algebra with $H_R(s) = 1 + es + (e-1)s^2$, then $\boldsymbol{L}_{1,e-1}(R)$ is an open subset of $\mathrm{M}_{e\times(e-1)}(k)$, and there are equalities*

$$\boldsymbol{L}_{1,e-1}(R) = \boldsymbol{P}^2_{1,e-1}(R) = \boldsymbol{L}^2_{1,e-1}(R)\,.$$

*Proof.* In view of (3.6.1), it suffices to show that if $\boldsymbol{L}^2_{1,e-1}(R)$ is not empty, then it is contained in $\boldsymbol{P}^2_{1,e-1}(R)$. Pick $C$ in $\boldsymbol{L}^2_{1,e-1}(R)$ and set $M = R^C$. One then has $H_M(s) = 1 + (e-1)s$, which implies $P^R_M(s,t) \equiv 1 + st + (st)^2 \pmod{t^3}$, by Lemma 1.10. Thus, for appropriate $a, b \in R_1$ there is an exact sequence

$$R(-2) \xrightarrow{b} R(-1) \xrightarrow{a} R \to M \to 0$$

It gives $bR = (0 : a)$ and hence $aR \subseteq (0 : (0 : a)) = (0 : b)$. The resulting relations

$$\operatorname{rank}_k(aR) \leq \operatorname{rank}_k(0 : b) = \operatorname{rank}_k(R/bR) = \operatorname{rank}_k(aR)$$

imply $(0 : b) = aR$, hence $bR \cong R/aR$. As a result, we obtain an exact sequence

$$0 \to M(-2) \to R(-1) \xrightarrow{a} R \to M \to 0$$ □

## 5. Algebras with Conca generators

In this section $R$ is a standard graded algebra, and we set

$$e = \operatorname{rank}_k R_1 \quad \text{and} \quad r = \operatorname{rank}_k R_2 \,.$$

A *Conca generator* of $R$ is a non-zero element $x \in R_1$ with $xR_1 = R_2$ and $x^2 = 0$, cf. [2]. We collect relevant facts about algebras containing such an element.

**5.1.** When $k$ is algebraically closed, generic quadratic algebras with $r \leq e-1$ have a Conca generator; see the proof of [5, Thm. 10].

**5.2.** If $x$ is a Conca generator for $R$, then one clearly has $r \leq e-1$ and $R_3 = 0$.

Furthermore, the algebra $R$ then is Koszul by [6, 2.7], see also [5, Lem. 2] or [2, 1.1], and every $R$-module $M$ with $xM = 0$ is linear by [2, 4.2].

**5.3.** The algebra $R$ has a Conca generator if and only if it has a presentation $R = k[x_1, \dots, x_e]/I$ with defining ideal $I$ generated by

$$x_l x_e \quad \text{for} \quad r+1 \leq l \leq e \,, \tag{5.3.1}$$

$$x_l x_{l'} - \sum_{h=1}^{r} a_{l,l';h} x_h x_e \quad \text{for} \quad 1 \leq l \leq l' \leq e-1 \,. \tag{5.3.2}$$

The class of $x_e$ is thus a Conca generator for $R$.

Theorem 2 from the introduction is contained in Propositions 5.4, 5.5, and 5.6. In their proofs, we use the order on $[1,e] \times [1,p]$ defined in (2.2.1).

**Proposition 5.4.** *Let $R$ be a standard graded algebra with a Conca generator.*

*For all positive integers $p,q$ with $q \leq (e-1)p$ one has $\boldsymbol{L}_{p,q}(R) \neq \varnothing$.*

*If $r = e-1$, then $\boldsymbol{L}_{p,q}(R) \neq \varnothing$ implies $q \leq (e-1)p$, one has $\boldsymbol{P}^2_{p,(e-1)p}(R) \neq \varnothing$ for each $p \geq 1$, the set $\boldsymbol{P}^2_{1,e-1}(R)$ is open in $\mathrm{M}_{1\times e-1}(k)$, and $\boldsymbol{P}^2_{1,e-1}(R) = \boldsymbol{L}_{1,e-1}(R)$.*

*Proof.* Let $\boldsymbol{s}$ be the set consisting of the $q$ smallest elements of $[1,e] \times [1,p]$, and $C \in \mathrm{M}_{ep\times q}(k)$ the matrix with $C_{\boldsymbol{s}}$ equal to the $q \times q$ unit matrix and $C_{(l,n)} = 0$ for $(l,n) \notin \boldsymbol{s}$. The condition $q \leq (e-1)p$ implies $(e,n) \notin \boldsymbol{s}$ for $n = 1, \dots, p$, hence $x_e R^C = 0$. Now recall that every $R$-module $M$ with $x_e M = 0$ is linear, see 5.2.

Assume $r = e-1$. When $\boldsymbol{L}_{p,q}(R) \neq \varnothing$, Corollary 1.7 yields $q \leq (e-1)p$. If $x$ is a Conca generator, a rank count gives $(0 : x) = xR$, so $(R/xR)^p$ is in $\boldsymbol{P}^2_{p,(e-1)p}(R)$. The sets $\boldsymbol{L}_{1,e-1}(R)$ and $\boldsymbol{P}^2_{1,e-1}(R)$ are equal and open by Proposition 4.5. □

**Proposition 5.5.** *Let $R$ be a standard graded algebra with a Conca generator.*

*If $q \leq e-1$, then the interior of $\boldsymbol{L}_{p,q}(R)$ is not empty.*

*Proof.* Assume first $r = e-1$. Now $R$ is Koszul, see 5.2, so $\boldsymbol{L}_{1,e-1}(R)$ is open and non-empty by Propositions 5.5 and 5.4. Using Theorem 3.2 we extend this conclusion first to $\boldsymbol{L}_{p,e-1}(R)$ for arbitrary $p$, then to all $\boldsymbol{L}_{p,q}(R)$ with $q \leq e-1$.

Assume next $r < e-1$. Set $R' = k[x_1, \dots, x_e]/I'$ where $I'$ is generated by $x_e^2$ and the polynomials in (5.3.2). One has $H_{R'}(s) = (1+(e-1)s)\cdot(1+s)$ and the class of $x_e$ in $R'$ is a Conca generator, so $R'$ is Koszul; see 5.2. The map $R' \to R$ is Golod; see the proof of [2, 3.2]. In $\mathrm{M}_{ep\times q}(k)$ this gives $\boldsymbol{L}_{p,q}(R) \supseteq \boldsymbol{L}_{p,q}(R')$, by Proposition 3.5, and $\boldsymbol{L}_{p,q}(R')$ has a non-empty interior by the case already settled. □

**Proposition 5.6.** *Let $R$ be a standard graded algebra with a Conca generator.*

*If $p,q \in \mathbb{N}$ satisfy $q \leq (e-r)p$, then $\boldsymbol{L}_{p,q}(R)$ contains the non-empty open set $\boldsymbol{L}^1_{p,q}(R) \cap \mathrm{M}_{ep\times q}(k)(\boldsymbol{s})$, where $\boldsymbol{s}$ consists of the $q$ largest elements of $[1,e] \times [1,p]$.*

*Proof.* Both $\mathrm{M}_{ep\times q}(k)(\boldsymbol{s})$ and $\boldsymbol{L}^1_{p,q}(R)$ are open and nonempty, see 2.2 and Lemma 3.7, so their intersection in the affine space $\mathrm{M}_{ep\times q}(k)$ has the same properties. It remains to prove that each $R$-module $M = R^C$ in this intersection is Koszul.

By definition, the rows of the matrix $C \in \mathrm{M}_{ep\times q}(k)(\boldsymbol{s})$ form a basis of the row space of $C$, so they determine elements $d_{(l,n),(l',n)} \in k$, such that

$$C_{(l,n)} = \sum_{(l',n')\in\boldsymbol{s}} d_{(l,n),(l',n')} C_{(l',n')} \quad \text{for all} \quad (l,n) \in [1,e]\times[1,p]\,.$$

In degrees 0 and 1 the following sequence of graded $R$-modules

$$\begin{array}{c} \qquad\qquad\qquad\qquad 1\otimes u_n \longmapsto u_n \\ R\otimes_k \bigoplus_{(l,n)\notin\boldsymbol{s}} (kx_l)\otimes_k (ku_n) \longrightarrow R\otimes_k \bigoplus_{n\in[1,p]} ku_n \longrightarrow M \longrightarrow 0 \\ \qquad 1\otimes x_l\otimes u_n \longmapsto x_l\otimes u_n - \sum_{(l',n')\in\boldsymbol{s}} d_{(l,n),(l',n')}(x_{l'}\otimes u_{n'}) \end{array}$$

is exact by construction, so it is exact because $M$ has a linear presentation.

The presentation of $M$ described above yields one for $S = R \ltimes M$, in the form

$$S \cong k[x_1,\dots x_e, y_1,\dots y_p]/J\,,$$

where $J$ is generated by the polynomials in (5.3.1), (5.3.2), and by those below:

$$\text{(5.6.1)} \qquad y_n y_{n'} \qquad \text{for} \quad 1\le n,n'\le p\,,$$

$$\text{(5.6.2)} \qquad x_l y_n - \sum_{(l',n')\in\boldsymbol{s}} d_{(l,n),(l',n')} x_l y_n \quad \text{for} \quad (l,n)\notin\boldsymbol{s}\,.$$

For monomials in $x_1,\dots,y_p$ we use the reverse degree-lexicographic order with

$$y_1 > \cdots > y_p > x_1 > \cdots > x_e\,.$$

Thus, $x_l y_n > x_{l'} y_{n'}$ is equivalent to $(l,n) < (l',n')$, so the choice of $\boldsymbol{s}$ implies:

$$x_l y_n > x_{l'} y_{n'} \quad \text{holds when} \quad (l,n)\notin\boldsymbol{s} \quad \text{and} \quad (l',n')\in\boldsymbol{s}\,.$$

For the chosen generators of $J$, this gives the following list of initial terms:

$$\begin{array}{llll} x_l x_{l'} & \text{for} \quad 1\le l\le l'\le e-1\,, & x_l y_n & \text{for} \quad (l,n)\notin\boldsymbol{s}\,, \\ x_l x_e & \text{for} \quad r+1\le l\le e\,, & y_n y_{n'} & \text{for} \quad 1\le n,n'\le p\,. \end{array}$$

Set $T = k[x_1,\dots,x_e,y_1,\dots,y_p]/L$, where $L$ denotes the ideal generated by the monomials listed above. We claim that $L$ contains all monomials of degree 3. Only those of the form $x_l x_{l'} y_n$ with $l\le l'$ need attention. Unless $l\le r$ and $l'=e$ hold, $x_l x_{l'}$ is in $L$, hence so is $x_l x_{l'} y_n$. For $l\le r$ the hypothesis $q\le(e-r)p$ and the choice of $\boldsymbol{s}$ imply $(l,e)\notin\boldsymbol{s}$, so $x_l y_n$ is $L$, hence so is $x_l x_e y_n$.

We just proved that $T_3$ is zero. Counting non-zero monomials in $T$, we get $H_T(s) = 1+(e+p)s+(e-1+q)s^2$. This gives $H_T(s) = H_S(s)$, which implies that the polynomials in (5.3.1), (5.3.2), and (5.6.1), (5.6.2) form a Gröbner basis for $S$. It follows that $S$ is Koszul, see [9, §4], hence so is $M$, by Proposition 1.5. $\square$

## 6. Open sets of linear modules

As before, $R$ denotes a standard graded $k$-algebra, and we set

$$e = \operatorname{rank}_k R_1 \quad \text{and} \quad r = \operatorname{rank}_k R_2\,.$$

We assume $e \geq 1$, and let $p \geq 1$ and $q \geq 0$ denote integers.

Here our goal is to record various instances when the linear locus $\boldsymbol{L}_{p,q}(R)$ is open and non-empty. The minimal admissible values of $r$ and $e$ are easily disposed of:

**6.1.** If $e = 1$ and $r = 1$, then $\boldsymbol{L}_{p,q}(R) = \varnothing$ for all $p$ and $q$.

Indeed, these conditions imply $R \cong k[x]$ or $R \cong k[x]/(x^n)$ for some $n \geq 3$.

**6.2.** When $e \geq 1$ and $r = 0$ $R$ is Koszul, $\boldsymbol{L}_{p,q}(R) = \boldsymbol{L}^0_{p,q}(R)$, and this set is open in $\mathrm{M}_{ep\times q}(k)$ for all $p$ and $q$; one has $\boldsymbol{L}_{p,q}(R) \neq \varnothing$ if and only if $q \leq ep$.

In view of the preceding remarks, we henceforth focus on the case $e \geq 2$.

The proofs of the next two propositions draw on most results in the paper.

**Proposition 6.3.** *If $e \geq 2$ and $R$ is short and Gorenstein, then it is Koszul.*

*When $e = 2$, for each pair $(p,q)$ one has $\boldsymbol{L}_{p,q}(R) = \boldsymbol{L}^{q-1}_{p,q}(R)$, this set is open in $\mathrm{M}_{ep\times q}(k)$, and $\boldsymbol{L}_{p,q}(R) \neq \varnothing$ if and only if $q \leq p$.*

*When $e \geq 3$, for each pair $(p,q)$ one has $\boldsymbol{L}_{p,q}(R) = \boldsymbol{L}^{m}_{p,q}(R)$ for some $m$, and this set is open in $\mathrm{M}_{ep\times q}(k)$; if there exists an non-zero element $x \in R_1$ with $x^2 = 0$ (in particular, if $k$ is algebraically closed), then $\boldsymbol{L}_{p,q}(R) \neq \varnothing$ for $q \leq (e-1)p$.*

*Proof.* For a proof that $R$ is Koszul see, for instance, [6, 2.7], or [2, 4.1].

The set $\boldsymbol{L}^m_{p,q}(R)$ is open in $\mathrm{M}_{ep\times q}(k)$ for all $m$, $p$, and $q$, see Lemma 3.7. In particular, the openness of $\boldsymbol{L}_{p,q}(R)$ follows from the other assertions.

For each $i \in \mathbb{Z}$, set $b_i = \beta^R_i(k)$ and $M_{(i)} = \operatorname{Hom}_R(\Omega^i_R(k), R)(1-i)$. One has:

(1) $b_i > b_{i-1}$ for every $i \geq 0$; moreover, $b_i = i+1$ when $e = 2$.

(2) $H_{M_{(i)}}(s) = b_{i-1} + b_i s$.

(3) If $N$ is an indecomposable non-Koszul module, then $N \cong M_{(i)}$ for some $i \geq 1$.

Indeed, the recurrence relation $b_{i+1} = eb_i - b_{i-1}$, for $i \geq 2$, given by (1.2.1), implies (1). Parts (2) and (3) are contained in [2, 4.6](2). Next we prove:

(4) $M_{(i)}$ is $(i-1)$-step linear, but not $i$-step linear.

Indeed, since $R$ is Koszul one obtains an exact sequence

$$0 \to \Omega^i_R(k) \to R(-i+1)^{b_{i-1}} \to \cdots \to R(-1)^{b_1} \to R \xrightarrow{\varepsilon} k \to 0\,,$$

from a minimal free resolution of $k$ over $R$. Now $\operatorname{Hom}_R(-,R)$ is exact because $R$ is Gorenstein, and $\operatorname{Hom}_R(k,R) \cong k(-2)$ as $R_2 \cong k$, so we get an exact sequence

$$0 \to k(-2) \xrightarrow{\eta} R \to R(1)^{b_1} \to \cdots \to R(i-1)^{b_{i-1}} \to \operatorname{Hom}_R(\Omega^i_R(k), R) \to 0$$

It gives for $M_{(i)}$ a minimal free resolution depicted below, which proves (4):

$$\cdots \to R(-i-1) \to R(-i+1) \to R(-i+2)^{b_1} \to \cdots \to R^{b_{i-1}} \to 0$$

Choose now, by (1), an integer $m$ so that $b_i > q$ holds for $i > m$; by the same token, pick $m = q-1$ when $e = 2$. If $\boldsymbol{L}^m_{p,q}(R)$ contains a module $M$ that is not Koszul, then some indecomposable direct summand $N$ of $M$ is not Koszul. By (3), we have $N \cong M_{(i)}$ for some $i \geq 1$, so $M_{(i)}$ is $m$-step linear. Now (4) implies $i > c$, hence $b_i > q$ by the choice of $m$. On the other hand, for the submodule $M_{(i)}$ of $M$ we get $b_i \leq q$ from (2). The contradiction implies $\boldsymbol{L}_{p,q}(R) = \boldsymbol{L}^m_{p,q}(R)$, as desired.

When $e=2$, Corollary 1.7 gives $\boldsymbol{L}_{p,q}(R)=\varnothing$ when $q>p$. Thus, we assume $q\le p$ and set out to prove $\boldsymbol{L}_{p,q}(R)\neq\varnothing$. In view of Lemma 3.4, we may restrict to the case $q=p$. Since $R$ is Koszul, we have $R\cong k[x,y]/(f,g)$ with $f,g$ a regular sequence in $Q_2$. Set $Q=k[x,y]/(f)$. Corollary 4.3 shows that it suffices to exhibit a non-zero-divisor $h\in Q_1$. If $f$ is irreducible, then $Q$ is an integral domain; take $h=x$. Else, $f$ is a product of two linear forms, so after a change of variables we may assume $(f)=(x^2)$ or $(f)=(xy)$; in either case, pick $h=x+y$.

Finally, as $R$ is Gorenstein every non-zero element $x\in R_1$ with $x^2=0$ evidently is a Conca generator; such an $x$ exists when $k$ is algebraically closed, see for instance, [5, Lem. 3]. Now Proposition 5.4 gives $\boldsymbol{L}_{p,q}(R)\neq\varnothing$ for $q\le(e-1)p$. □

**Proposition 6.4.** *Assume that $R$ is quadratic, with $H_R(s)=1+es+(e-1)s^2$. If $e=2$, or if $e=3$ and $k$ is infinite, then the following hold.*

(1) *There is an isomorphism $R\cong Q/(g)$ for a Koszul $k$-algebra $Q$ and a non-zero-divisor $g\in Q_2$; in particular, $R$ is Koszul.*
(2) *For every positive integer $p$ one has $\boldsymbol{L}_{p,(e-1)p}(R)=\boldsymbol{P}^2_{p,(e-1)p}(R)$, this set is open in $\mathrm{M}_{ep\times(e-1)p}(k)$, and is not empty.*

*Proof.* (1) When $e=2$ one has $R\cong k[x,y]/(f,g)$ with $f,g$ a regular sequence of quadrics; take $Q=k[x,y]/(f)$.

When $e=3$, write $R\cong k[x,y,z]/I$ with $I$ an ideal minimally generated by 4 quadrics. Assuming $hR_1=0$ for some $h\in R_1$ with $h\neq 0$, we get a quadratic algebra $S=R/(h)$ with $H_S(s)=1+2s+2s^2$; this is impossible, and so we get $(0:\mathfrak{m})=R_2$. Thus, $R$ is an *almost complete intersection of codimension* 3 *and type* 2. In the local case, such rings are described by a structure theorem of Buchsbaum and Eisenbud, see [4, 5.4]. This is a corollary of [4, 5.3], whose proof refers to a general position argument to find generators $f_1,f_2,f_3,f_4$ of $I$, every 3 of which form a regular sequence. The hypothesis that $k$ is infinite allows one to find the $f_i$ as $k$-linear combinations of the original quadrics. The rest of the proof of [4, 5.3] and that of [4, 5.4] now yield $\{i_1,i_2,i_3\}\subset[1,4]$ and $f\in Q_2$, such that $I=(f_{i_1},f_{i_2},f_{i_3},f)$ and $f_{i_1}$ is a non-zero-divisor on the algebra $Q=k[x,y,z]/(f_{i_2},f_{i_3},f)$.

(2) In view of (1), Corollary 4.3 applies. It yields $\boldsymbol{L}_{p,(e-1)p}(R)=\boldsymbol{P}^2_{p,(e-1)p}(R)$, shows that this set is open in $\mathrm{M}_{ep\times(e-1)p}(k)$, and also that it is non-empty when $e=3$. When $e=2$, we get $\boldsymbol{L}_{p,p}(R)\neq\varnothing$ from Proposition 6.3. □

To finish, we show that when $R$ needs more that 3 generators, there exists no similar description of the locus of modules with constant Betti numbers, and we isolate the smallest case, when it is not known whether this set has an open interior.

*Remark* 6.5. Choose an element $a\in k\smallsetminus\{0,\pm1\}$ and set $R=k[x_1,x_2,x_3,x_4]/I$, where $I$ is the ideal generated by the following quadratic forms in four variables:

$$x_1^2\,,\quad ax_1x_3+x_2x_3\,,\quad x_1x_4+x_2x_4\,,\quad x_2^2\,,\quad x_3^2\,,\quad x_3x_4\,,\quad x_4^2\,.$$

Since $x_4$ is a Conca generator for $R$, Proposition 5.4 shows that the sets $\boldsymbol{L}_{1,3}(R)$ and $\boldsymbol{P}^2_{1,3}(R)$ are equal, open, and non-empty. On the other hand, [11, 3.4] and Proposition 5.4 give, respectively, the strict inclusion and the inequality below:

$$\boldsymbol{L}_{2,6}(R)\supsetneq\boldsymbol{P}^2_{2,6}(R)\neq\varnothing\,.$$

We do not know whether either set above has a non-empty interior in $\mathrm{M}_{8\times 6}(k)$.

## References

[1] L. L. Avramov, *Infinite free resolutions*, Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math. **166**, Birkhäuser, Basel, 1998; pp. 1–118.

[2] L. L. Avramov, S. B. Iyengar, L. M. Şega, *Free resolutions over short local rings*, J. London Math. Soc. **78** (2008), 459–476.

[3] J. Backelin, R. Fröberg, *Koszul algebras, Veronese subrings and rings with linear resolutions*, Rev. Roumaine Math. Pures Appl. **30** (1985), 85–97.

[4] D. A. Buchsbaum, D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension* 3, Amer. J. Math. **99** (1977), 447–485.

[5] A. Conca, *Gröbner bases for spaces of quadrics of low codimension*, Adv. Appl. Math. **24** (2000), 111–124.

[6] A. Conca, M.-E. Rossi, G. Valla, *Gröbner flags and Gorenstein algebras*, Compositio Math. **129** (2000), 95–121.

[7] A. Conca, N. V. Trung, G. Valla, *Koszul property for points in projective spaces*, Math. Scand. **89** (2001), 201–216.

[8] D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260** (1980), 35–64.

[9] R. Fröberg, *Koszul algebras*, Advances in commutative ring theory (Fez, 1997), 337–350, Lecture Notes in Pure and Appl. Math. **205**, Dekker, New York, 1999

[10] R. Fröberg, C. Löfwall, *Koszul homology and Lie algebras with application to generic forms and points*, Homology, Homotopy, Appl. **4:2** (2002), 227–258.

[11] V. N. Gasharov, I. V. Peeva, *Boundedness versus periodicity over commutative local rings*, Trans. Amer. Math. Soc. **320** (1990), 569–580.

[12] A. Grothendieck, J. Dieudonné, *Éléments de géometrie algébrique, IV. Étude locale des schémas et des morphismes de schémas (Seconde partie)*, Publ. Math. IHES **24**, 1965.

[13] A. Grothendieck, J. Dieudonné, *Éléments de géometrie algébrique, IV. Étude locale des schémas et des morphismes de schémas (Troisième partie)*, Publ. Math. IHES **28**, 1966.

[14] T. H. Gulliksen, *Massey operations and the Poincaré series of certain local rings*, J. Algebra **22** (1972), 223–232.

[15] D. A. Jorgensen, L. M. Şega, *Nonvanishing cohomology and classes of Gorenstein rings*, Adv. Math. **188** (2004), 470–490.

[16] D. A. Jorgensen, L. M. Şega, *Independence of the total reflexivity conditions for modules*, Algebr. Represent. Theory **9** (2006), 217–226.

[17] J. Lescot, *Asymptotic properties of Betti numbers of modules over certain rings*, J. Pure Appl. Algebra **38** (1985), 287–298.

[18] C. Löfwall, *On the subalgebra generated by the one-dimensional elements in the Yoneda Ext-algebra*, Algebra, algebraic topology and their interactions (Stockholm, 1983), Lecture Notes in Math. **1183**, Springer, Berlin, 1986; pp. 291–338.

[19] A. Polishchuk, L. Positselski, *Quadratic algebras*, Univ. Lecture Ser. **37**, Amer. Math. Soc., Providence, RI, 2005.

[20] J.-E. Roos, *Relations between Poincaré-Betti series of loop spaces and of local rings*, Séminaire d'Algèbre Paul Dubreil, 31ème année (Paris, 1977–1978), Lecture Notes in Math. **740**, Springer, Berlin, 1979; pp. 285–322.

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