SOME PROPERTIES OF GRADED LOCAL COHOMOLOGY MODULES

CHRISTEL ROTTHAUS AND LIANA M. ŞEGA

Abstract. We consider a finitely generated graded module M over a standard graded commutative Noetherian ring $R = \bigoplus_{d \geq 0} R_d$ and we study the local cohomology modules $H^i_{R_+}(M)$ with respect to the irrelevant ideal R_+ of R. We prove that the top nonvanishing local cohomology is tame, and the set of its minimal associated primes is finite. When M is Cohen-Macaulay and R_0 is local, we establish new formulas for the index of the top, respectively bottom, nonvanishing local cohomology. As a consequence, we obtain that the (S_k)-loci of a Cohen-Macaulay R-module M, regarded as an R_0-module, are open in $\operatorname{Spec}(R_0)$. Also, when $\dim(R_0) \leq 2$ and M is a Cohen-Macaulay R-module, we prove that $H^i_{R_+}(M)$ is tame, and its set of minimal associated primes is finite for all i.

INTRODUCTION

Let $R = \bigoplus_{d \geq 0} R_d$ be a positively graded commutative Noetherian ring which is standard in the sense that $R = R_0[R_1]$, and set $R_+ = \bigoplus_{d>0} R_d$, the irrelevant ideal of R. Let $M = \bigoplus_{d \in \mathbb{Z}} M_d$ be a finitely generated graded R-module. In this paper we study the graded local cohomology modules $H^i_{R_+}(M)$.

It is known that each of the graded components $H^i_{R_+}(M)_j$ is finitely generated over R_0 and $H^i_{R_+}(M)_j = 0$ for all $j \gg 0$. Brodmann and Hellus [3] have recently raised the question whether the modules $H^i_{R_+}(M)$ are *tame* (or *asymptotically gap free*), meaning that either $H^i_{R_+}(M)_j = 0$ for all $j \ll 0$ or $H^i_{R_+}(M)_j \neq 0$ for all $j \ll 0$. A positive answer is known in several cases, cf. [1], [3], [9], [12], [13].

In order to understand the finiteness properties of the modules $H^i_{R_+}(M)$, Huneke [7] asked whether the set of their associated primes is finite; this was answered negatively by Singh [17]. However, as noted by Katzman in [8], it is not known whether the set of minimal associated primes is finite, or in other words, whether the support of $H^i_{R_+}(M)$ is Zariski-closed.

When i is the index of the bottom nonvanishing local cohomology, it is known that the set $\operatorname{Ass}_R\left(H^i_{R_+}(M)\right)$ is finite, and it follows that $H^i_{R_+}(M)$ is tame, cf [3]. When i is the index of the top nonvanishing local cohomology and $M = R$, it is proved in [9] that $H^i_{R_+}(R)$ has only finitely many minimal associated primes. In this paper we prove:

Theorem 1. *If $H^n_{R_+}(M) \neq 0$ and $H^i_{R_+}(M) = 0$ for all $i > n$, then:*

(1) *$H^n_{R_+}(M)$ is tame*
(2) *$H^n_{R_+}(M)$ has finitely many minimal associated primes.*

Date: June 3, 2004.

To study local cohomology for all indices i, some particular cases are treated.

Theorem 2. *Assume that M is a Cohen-Macaulay R-module, and either $\dim(R_0) \leq 2$ or $\dim(R_0) \leq 3$ and R_0 is semilocal. The modules $H^i_{R_+}(M)$ have then finitely many minimal associated primes for all i.*

When R_0 is semilocal of dimension at most 2, Brodmann, Fumasoli and Lim [1] proved that $H^i_{R_+}(M)$ is tame for all i. Assuming that M is Cohen-Macaulay, we eliminate the condition that R_0 is semilocal. This recovers a result of Lim [13].

Theorem 3. *If M is a Cohen-Macaulay R-module and $\dim(R_0) \leq 2$, then $H^i_{R_+}(M)$ is tame for all i.*

When M is a Cohen-Macaulay R-module and R_0 is local we prove:

$$\sup\{i \mid H^i_{R_+}(M) = 0 \text{ for all } j < i\} = \dim_R M - \dim_{R_0} M$$
$$\inf\{i \mid H^i_{R_+}(M) = 0 \text{ for all } j > i\} = \dim_R M - \operatorname{depth}_{R_0} M$$

Note that the notion of depth of M over R_0 is meaningful, cf. [16] for details. In particular, when R_0 is local and M is Cohen-Macaulay over R, the following statements are equivalent:

(a) $\operatorname{depth}_{R_0}(M) = \dim_{R_0}(M)$
(b) M_i is a Cohen-Macaulay R_0-module with $\dim_{R_0}(M_i) = \dim_{R_0}(M)$ for all i.
(c) There exists $j \geq 0$ so that $H^i_{R_+}(M) = 0$ for all $i \neq j$.

Another consequence of the formulas is somewhat surprising. Recall that a finite module N over a commutative Noetherian ring A satisfies the *Serre condition* (S_k) if $\operatorname{depth}_{A_\mathfrak{p}}(N_\mathfrak{p}) \geq \min\{k, \dim N_\mathfrak{p}\}$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$. When the ring A is excellent, Grothendieck [6] proved that the set

$$U_{S_k}(N) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \text{the } A_\mathfrak{p}\text{-module } N_\mathfrak{p} \text{ satisfies } (S_k)\}$$

is open in $\operatorname{Spec}(A)$. Noting that the above notions make sense for $A = R_0$ and $N = M$, the authors extended this result in [16] and proved that the set

$$U^0_{S_k}(M) = \{\mathfrak{p} \in \operatorname{Spec}(R_0) \mid \text{the } (R_0)_\mathfrak{p}\text{-module } M_\mathfrak{p} \text{ satisfies } (S_k)\}$$

is open in $\operatorname{Spec}(R_0)$ whenever R is excellent. In general, the condition that the ring is excellent is necessary for the (S_k)-loci to be open. However, when M is Cohen-Macaulay, it can be removed:

Theorem 4. *If M is Cohen-Macaulay over R, then for any $k \in \mathbb{N}$ the set $U^0_{S_k}(M)$ is open in $\operatorname{Spec}(R_0)$.*

The paper is organized as follows: In the first section we collect definitions and several known results on local cohomology that are used throughout the paper.

In the second section we prove Theorem 1(1) as Theorem 2.8. A stronger result is obtained when R_0 is local, with maximal ideal $\mathfrak{m}_0$. In this case, we prove that if n denotes the largest integer i with $H^i_{R_+}(M) \neq 0$, then the R-module $H^n_{R_+}(M)/\mathfrak{m}_0 H^n_{R_+}(M)$ is Artinian; in particular, this shows that the minimal number of generators of $H^n_{R_+}(M)_j$ has polynomial growth for $j \ll 0$.

In the third section we prove that certain subsets of $\operatorname{Spec}(R_0)$ are open. In particular, Theorem 1(2) is proved as Theorem 3.5. (See also 1.1.)

In Section 4 we obtain the formulas above for the top and bottom nonvanishing local cohomology, and prove Theorem 4 as Corollary 4.9.

In Section 5 we prove Theorem 2 as Theorems 5.3 and 5.4, and Theorem 3 as Theorem 5.6.

1. PRELIMINARIES

Throughout the whole paper, we let R denote a positively graded commutative Noetherian ring $R = \bigoplus_{d\geq 0} R_d$, which is standard in the sense that $R = R_0[R_1]$, and set $R_+ = \bigoplus_{i>0} R_i$, the irrelevant ideal of R. Let $M = \bigoplus_{d\in\mathbb{Z}} M_d$ be a finitely generated graded R-module.

For any M as above and any integer a we let $M[a]$ denote the a-shift of M, defined as the graded R-module with $M[a]_i = M_{i-a}$.

1.1. As noted in [3, 5.5] there is a bijection between the sets $\mathrm{Ass}_R\left(H^i_{R_+}(M)\right)$ and $\mathrm{Ass}_{R_0}\left(H^i_{R_+}(M)\right)$ given by $\mathfrak{p} + R_+ \mapsto \mathfrak{p}$. Since over a Noetherian ring the set of minimal primes of the support of a module coincides with the set of minimal associated primes of the module, the following statements are equivalent:

(a) The R-module $H^i_{R_+}(M)$ has finitely many minimal associated primes.
(b) The R_0-module $H^i_{R_+}(M)$ has finitely many minimal associated primes.
(c) The set $\mathrm{Supp}_R\left(H^i_{R_+}(M)\right)$ is closed in $\mathrm{Spec}(R)$.
(d) The set $\mathrm{Supp}_{R_0}\left(H^i_{R_+}(M)\right)$ is closed in $\mathrm{Spec}(R_0)$.

1.2. We set

$$g_R(M) = \mathrm{grade}(R_+, M)$$

Recall that $\mathrm{grade}(R_+, M) = \infty$ if and only if $R_+M = M$. In our case, this is equivalent to $M = 0$.

The following relation is known, cf. [4, 6.2.7]:

$$g_R(M) = \sup\{i \mid H^j_{R_+}(M) = 0 \text{ for all } j < i\} \tag{1.2.1}$$

When R_0 is local with maximal ideal $\mathfrak{m}_0$ we set

$$n_R(M) = \dim_R(M/\mathfrak{m}_0 M)$$

We make the convention that $\dim_R(0) = -\infty$, and note that $n_R(M) = -\infty$ if and only if $M = 0$. By [3, 3.4], we have

$$n_R(M) = \inf\{i \mid H^j_{R_+}(M) = 0 \text{ for all } j > i\} \tag{1.2.2}$$

1.3. We recall the following properties (see for example [4]) :

1.3.1. (Homogeneous Prime Avoidance Lemma) If $P_1, \dots, P_s$ are prime ideals in $\mathrm{Spec}(R)$ and R_+ is not contained in P_i for all i, then there exists a homogeneous element in $R_+ \setminus (P_1 \cup \dots \cup P_s)$.

In particular, if $0 < g_R(M) < \infty$, then there exists a homogeneous element $x \in R_+$ which is M-regular.

1.3.2. $H^i_{R_+}(M) \cong H^i_{R_+}\left(M/\Gamma_{R_+}(M)\right)$ for all $i > 0$.

1.3.3. (Flat Base Change) If $R \to R'$ is a flat homomorphism of Noetherian rings, then $H^i_{R_+}(M) \otimes_R R' \cong H^i_{R_+R'}(M')$ for all i, where $M' = M \otimes_R R'$.

In particular, if $H^i_{R_+R_\mathfrak{q}}(M_\mathfrak{q}) = 0$ for some $\mathfrak{q} \in \mathrm{Spec}(R_0)$, then $H^i_{R_+R_\mathfrak{p}}(M_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R_0)$ with $\mathfrak{p} \subseteq \mathfrak{q}$.

1.3.4. (Independence Theorem) If $R \to R'$ is a homomorphism of Noetherian rings and N is a finite R'-module (with the induced structure of R-module) then $H^i_{R_+R'}(N) \cong H^i_{R_+}(N)$ for all i.

Moreover, when R_0 is local with maximal ideal $\mathfrak{m}_0$ we have, cf. [3]:

1.3.5. $n_R(M) \leq 0$ if and only if $M = \Gamma_{R_+}(M)$.

1.3.6. $n_R(M) = n_R\big(M/\Gamma_{R_+}(M)\big)$, provided that $n_R(M) > 0$.

1.3.7. If $n_R(M) > 0$, then there exists a homogeneous element $x \in R_+$ such that $n_R(M/xM) = n_R(M) - 1$. This follows from 1.3.1, by choosing x to avoid all the minimal primes of $(\mathfrak{m}_0 M :_R M)$. Moreover, if $g_R(M) > 0$, then the element x may be chosen to be also M-regular.

2. The top local cohomology is tame

The assumptions on R and M are as in the first section.

2.1. **Theorem.** *Assume that R_0 is local with maximal ideal $\mathfrak{m}_0$.*
If $n = n_R(M)$, then the R-module $H^n_{R_+}(M)/\mathfrak{m}_0 H^n_{R_+}(M)$ is Artinian.

Proof. We will prove the statement by induction on $n_R(M)$.

Assume that $n_R(M) = 0$. By 1.3.5, we have then $M = \Gamma_{R_+}(M)$. It follows that $\operatorname{Supp}_R(M/\mathfrak{m}_0 M) = \{\mathfrak{m}_0 + R_+\}$. Since M is finitely generated as an R-module, we conclude that that $M/\mathfrak{m}_0 M$ is Artinian.

Now assume that we proved the statement for any finitely generated graded R-module N with $n_R(N) = k - 1 \geq 0$. We want to prove it for $n_R(M) = k$. In view of 1.3.2 and 1.3.6 we may replace M with $M/\Gamma_{R_+}(M)$, so that we may assume $\Gamma_{R_+}(M) = 0$. Let $x \in R_+$ be a homogeneous M-regular element with $\deg(x) = a$ such that $n_R(M/xM) = n_R(M) - 1 = k - 1$ (see 1.3.7).

The short exact sequence

$$0 \to M \xrightarrow{x} M[-a] \to \big(M/xM\big)[-a] \to 0$$

yields a long exact sequence

$$\cdots \to H^{k-1}_{R_+}(M/xM) \to H^k_{R_+}(M) \xrightarrow{x} H^k_{R_+}(M)[-a] \to H^k_{R_+}(M/xM)[-a] \to \cdots$$

Since $n_R(M/xM) = k - 1$, we have $H^k_{R_+}(M/xM) = 0$ by (1.2.2). Let L denote the kernel of the multiplication by x on $H^k_{R_+}(M)$. The induction hypothesis yields that the R-module $H^{k-1}_{R_+}(M/xM)/\mathfrak{m}_0 H^{k-1}_{R_+}(M/xM)$ is Artinian. As a homomorphic image of this module, $L/\mathfrak{m}_0 L$ is also Artinian.

We have then an exact sequence

$$L/\mathfrak{m}_0 L \to H^k_{R_+}(M)/\mathfrak{m}_0 H^k_{R_+}(M) \xrightarrow{x} H^k_{R_+}(M)/\mathfrak{m}_0 H^k_{R_+}(M)[-a] \to 0$$

which shows that the kernel of multiplication by x on $H^k_{R_+}(M)/\mathfrak{m}_0 H^k_{R_+}(M)$ is an Artinian R-module. Since $H^k_{R_+}(M)/\mathfrak{m}_0 H^k_{R_+}(M)$ is an (x)-torsion R-module, we conclude that it is Artinian using for example a result of Melkersson [15, 1.3]. □

2.2. Recall from [4, 15.1.5] that for all i and n the R_0-module $H^n_{R_+}(M)_i$ is finitely generated. When $(R_0, \mathfrak{m}_0)$ is local, it makes thus sense to introduce the numbers

$$\ell^n_R(M)_i := \operatorname{length}_{R_0}\big(H^n_{R_+}(M)/\mathfrak{m}_0 H^n_{R_+}(M)\big)_i$$

Recall also that $H^n_{R_+}(M)_i = 0$ for $i \gg 0$.

2.3. Since R is finitely generated (in degree 1), it is isomorphic to a quotient of a polynomial ring $S = R_0[x_1, \dots x_s]$, with variables in degree 1. By 1.3.4, we have $H^i_{R_+}(M) \cong H^i_{S_+}(M)$ for all i.

When $(R_0, \mathfrak{m}_0)$ is local, it follows from 1.2.2 that $n_R(M) = n_S(M)$. Note that for every i and n we also have $\ell^n_R(M)_i = \ell^n_S(M)_i$.

2.4. **Corollary.** *Set $n = n_R(M)$ and assume that R is generated over R_0 by s elements. There exists then a polynomial $q(t) \in \mathbb{Q}[t]$ of degree at most s such that*

$$\ell^n_R(M)_i = q(i) \quad \textit{for all} \quad i \ll 0\,.$$

Proof. By 2.3 we may assume $R = R_0[x_1, \dots x_s]$. The existence of the polynomial $q(t)$ is given for example by [10, 2], using Theorem 2.1. □

We recall a terminology introduced in [3]: A graded R-module $T = \oplus_{d\in\mathbb{Z}} T_d$ is said to be *tame* (or *asymptotically gap free*) if the set

$$\{d \in \mathbb{Z} \mid T_d \neq 0, T_{d+1} = 0\}$$

is finite. Clearly, all Artinian and Noetherian R-modules are tame. Theorem 2.1 shows thus that the R-module $H^n_R(M)/\mathfrak{m}_0 H^n_R(M)$ is tame when $n = n_R(M)$. Using Nakayama's Lemma, we note:

2.5. **Corollary.** *If $n = n_R(M)$, then $H^n_{R_+}(M)$ is tame.* □

We note that the top local cohomology module is almost never Noetherian:

2.6. *Remark.* Assume that R_0 is local and set $n = n_R(M)$. If $n > 0$, then $H^n_{R_+}(M)_j \neq 0$ for all $j \ll 0$.

Indeed, by 1.3.2, we may assume $\Gamma_{R_+}(M) = 0$ and thus $g_R(M) > 0$. By 1.3.7 there exists then a homogeneous M-regular element $x \in R_+$ with $\deg(x) = a$ such that $n_R(M/xM) = n_R(M) - 1$. In particular, this implies that $H^n_{R_+}(M/xM) = 0$. The long exact sequence in homology

$$\cdots \to H^n_{R_+}(M) \xrightarrow{x} H^n_{R_+}(M)[-a] \to H^n_{R_+}(M/xM)[-a] \to \cdots$$

shows that multiplication by x on $H^n_{R_+}(M)$ is surjective, hence there exist infinitely many indices j with $H^n_{R_+}(M)_j \neq 0$. In view of Corollary 2.5, we have $H^n_{R_+}(M)_j \neq 0$ for all $j \ll 0$. (Alternatively, this result can be proved by reducing to the case when the residue field of R_0 is infinite, in which case we may assume $a = 1$.)

For the rest of the section we remove the assumption that R_0 is local.

2.7. *Remark.* Let n be an integer such that $H^i_{R_+}(M) = 0$ for all $i > n$ and $H^n_{R_+}(M) \neq 0$. Using (1.2.2) and 1.3.3 we see that $n_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq n$ for all $\mathfrak{p} \in \operatorname{Spec}(R_0)$. In particular, this implies that $H^n_{R_+R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$ if and only if $n = n_{R_\mathfrak{p}}(M_\mathfrak{p})$. If $n \neq n_{R_\mathfrak{p}}(M_\mathfrak{p})$ for all $\mathfrak{p}$, then $H^n_{R_+R_\mathfrak{p}}(M_\mathfrak{p}) = 0$ for all $\mathfrak{p}$, hence $H^n_{R_+}(M) = 0$, a contradiction. In conclusion, there exists some $\mathfrak{p} \in \operatorname{Spec}(R_0)$ such that $n = n_{R_\mathfrak{p}}(M_\mathfrak{p})$.

2.8. **Theorem.** *If n is an integer such that $H^i_{R_+}(M) = 0$ for all $i > n$ and $H^n_{R_+}(M) \neq 0$, then $H^n_{R_+}(M)$ is tame.*

Proof. If $n = 0$, then $H^n_{R_+}(M)$ is a finite R_0-module, and thus $H^n_{R_+}(M)_j = 0$ for all $j \ll 0$. So we may assume $n > 0$.

By Remark 2.7, there exists $\mathfrak{p} \in \operatorname{Spec}(R_0)$ such that $n = n_{R_\mathfrak{p}}(M_\mathfrak{p})$. By Remark 2.6, it follows that $H^n_{R_+R_\mathfrak{p}}(M_\mathfrak{p})_j = 0$ for all $j \ll 0$, and thus $H^n_{R_+}(M) = 0$ for all $j \ll 0$. □

3. THE SUPPORT OF THE TOP LOCAL COHOMOLOGY IS CLOSED

In this section we prove several results regarding open loci, which culminate with the one announced in the title. First, we record a basic lemma:

3.1. Lemma. *Let $N = \bigoplus_{d\in\mathbb{Z}} N_d$ be a finitely generated R-module and $\mathfrak{p} \in \operatorname{Spec}(R_0)$ be a prime ideal.*

(1) *If $N_\mathfrak{p} = 0$ then there exists an open set $\mathcal{U}$ of $\operatorname{Spec}(R_0)$ such that $\mathfrak{p} \in \mathcal{U}$ and $N_\mathfrak{q} = 0$ for all $\mathfrak{q} \in \mathcal{U}$.*
(2) *Let $x \in R_+$ be a homogeneous element such that $x/1 \in R_+R_\mathfrak{p}$ is $N_\mathfrak{p}$-regular. There exists then an open set $\mathcal{U}$ of $\operatorname{Spec}(R_0)$ such that $\mathfrak{p} \in \mathcal{U}$ and $N_\mathfrak{q} = 0$ or $x/1 \in R_+R_\mathfrak{q}$ is N_q-regular for al $q \in \mathcal{U}$.*

Proof. (1) Let $n_1, \dots n_s$ be a set of generators of N over R and let $a \in R_0 \smallsetminus \mathfrak{p}$ such that $n_1/1, \dots, n_s/1 = 0$ in N_a. Consider then

$$\mathcal{U} = U_a := \{\mathfrak{p} \in \operatorname{Spec}(R_0) \mid a \notin \mathfrak{p}\}\,.$$

(2) Let K be the kernel of multiplication by x on N. Since $x/1$ is $N_\mathfrak{p}$-regular, we have $K_\mathfrak{p} = 0$. Use then (1) to find an open set $\mathcal{U}$ such that $K_\mathfrak{q} = 0$ for all $\mathfrak{q} \in \mathcal{U}$. □

3.2. Lemma. *For any $\mathfrak{q} \in \operatorname{Spec}(R_0)$ we set $n_\mathfrak{q} = n_{R_\mathfrak{q}}(M_\mathfrak{q})$.*

(1) *If $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals in $\operatorname{Spec}(R_0)$ such that $\mathfrak{p} \subseteq \mathfrak{q}$, then $n_\mathfrak{p} \le n_\mathfrak{q}$.*
(2) *For any $\mathfrak{p} \in \operatorname{Spec}(R_0)$ there exists an open set $\mathcal{U} \subseteq \operatorname{Spec}(R_0)$ such that $\mathfrak{p} \in \mathcal{U}$ and $n_\mathfrak{q} \le n_\mathfrak{p}$ for all $\mathfrak{q} \in \mathcal{U}$; in particular, $n_\mathfrak{q} = n_\mathfrak{p}$ for all $\mathfrak{q} \in \mathcal{U} \cap V(\mathfrak{p})$.*

Proof. (1) By localizing at $\mathfrak{q}$, we may assume that R_0 is local, with maximal ideal $\mathfrak{q}$, and hence $n_\mathfrak{q} = n_R(M)$. If $n_\mathfrak{p} = -\infty$, then the inequality is clear. Assume now that $n_\mathfrak{p} \ge 0$. From (1.2.2) we have $H^{n_\mathfrak{p}}_{R_+R_\mathfrak{p}}(M_\mathfrak{p}) \ne 0$ and from 1.3.3 it follows that $H^{n_\mathfrak{p}}_{R_+}(M) \ne 0$. Using again (1.2.2), we conclude $n_\mathfrak{p} \le n_R(M) = n_\mathfrak{q}$.

(2) We proceed by induction on $n_\mathfrak{p}$. If $n_\mathfrak{p} = -\infty$, then $M_\mathfrak{p} = 0$ and we choose then $\mathcal{U}$ as in Lemma 3.1(1) so that $\mathfrak{p} \in \mathcal{U}$ and for all $\mathfrak{q} \in \mathcal{U}$ we have $M_\mathfrak{q} = 0$, and thus $n_\mathfrak{q} = -\infty$.

If $n_\mathfrak{p} = 0$ then $\big(M/\Gamma_{R_+}(M)\big)_\mathfrak{p} = 0$ by 1.3.5. Using Lema 3.1(1) we choose then $\mathcal{U}$ so that $\mathfrak{p} \in \mathcal{U}$ and $\big(M/\Gamma_{R_+}(M)\big)_\mathfrak{q} = 0$ for all $q \in \mathcal{U}$. For such $\mathfrak{q}$ it follows that $n_\mathfrak{q} \le 0$ using again 1.3.5.

Assume now that $n_\mathfrak{p} = n > 0$ and that the statement is proved for all graded finitely generated R modules N with $n_{R_\mathfrak{p}}(N) = n-1$. Set $\overline{M} = M/\Gamma_{R_+}(M)$. Since $n_\mathfrak{p} > 0$, we have $n_\mathfrak{p} = n_{R_\mathfrak{p}}(\overline{M}_\mathfrak{p})$ by 1.3.6. Let $x \in R_+$ be a homogeneous $\overline{M}_\mathfrak{p}$-regular element such that $n_{R_\mathfrak{p}}(\overline{M}_\mathfrak{p}/x\overline{M}_\mathfrak{p}) = n_\mathfrak{p} - 1$, cf. 1.3.7. By the induction hypothesis, there exists an open set $\mathcal{U}$ such that $\mathfrak{p} \in \mathcal{U}$ and $n_{R_\mathfrak{p}}(\overline{M}_\mathfrak{p}/x\overline{M}_\mathfrak{p}) \ge n_{R_\mathfrak{q}}(\overline{M}_\mathfrak{q}/x\overline{M}_\mathfrak{q})$ for all $\mathfrak{q} \in \mathcal{U}$. The inequality in the statement is clearly satisfied for all $\mathfrak{q} \in \mathcal{U}$ with $n_\mathfrak{q} \le 0$. For all $\mathfrak{q} \in \mathcal{U}$ with $n_\mathfrak{q} > 0$ we have:

$$n_\mathfrak{p} = n_{R_\mathfrak{p}}(\overline{M}_\mathfrak{p}/x\overline{M}_\mathfrak{p}) + 1 \ge n_{R_\mathfrak{q}}(\overline{M}_\mathfrak{q}/x\overline{M}_\mathfrak{q}) + 1 \ge n_\mathfrak{q}(\overline{M}_\mathfrak{q}) = n_\mathfrak{q}(M_\mathfrak{q})$$

For the second inequality, note that if S is a graded ring with unique graded maximal ideal $\mathfrak{n}$, then for any nonzero finitely generated graded S-module N, and any homogeneous element $z \in \mathfrak{n}$ we have $\dim_R(N/zN) \ge \dim_R(N) - 1$; this is the graded version of [5, A.4]. To prove this, it suffices to reduce the problem to the local case, using the fact $\dim_R(N) = \dim_{R_\mathfrak{n}}(N_\mathfrak{n})$ and $\dim_R(N/zN) = \dim_{R_\mathfrak{n}}(N_\mathfrak{n}/zN_\mathfrak{n})$. □

We obtain a similar lemma for the grade:

3.3. **Lemma.** *For any* $\mathfrak{p} \in \operatorname{Spec}(R_0)$ *we set* $g_{\mathfrak{p}} = g_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$. .

(1) *If* $\mathfrak{p}$, $\mathfrak{q}$ *are prime ideals in* $\operatorname{Spec}(R_0)$ *such that* $\mathfrak{p} \subseteq \mathfrak{q}$, *then* $g_{\mathfrak{p}} \geq g_{\mathfrak{q}}$.
(2) *For any* $\mathfrak{p} \in \operatorname{Spec}(R_0)$ *there exists an open set* $\mathcal{U} \subseteq \operatorname{Spec}(R_0)$ *such that* $\mathfrak{p} \in \mathcal{U}$ *and* $g_{\mathfrak{q}} \geq g_{\mathfrak{p}}$ *for all* $\mathfrak{q} \in \mathcal{U}$. *In particular,* $g_{\mathfrak{q}} = g_{\mathfrak{p}}$ *for all* $\mathfrak{q} \in \mathcal{U} \cap V(\mathfrak{p})$.

Proof. (1) By localizing at $\mathfrak{q}$ we may assume that R_0 is local with maximal ideal $\mathfrak{q}$ and thus $g_{\mathfrak{q}} = g_R(M)$. If $g_{\mathfrak{p}} = \infty$, then the inequality is clear. Assume now that $g_{\mathfrak{p}} \neq \infty$. By (1.2.1) we have $H^{g_{\mathfrak{p}}}_{R_+R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0$. From 1.3.3 it follows that $H^{g_{\mathfrak{p}}}_{R_+}(M) \neq 0$. Using again (1.2.1) we conclude $g_{\mathfrak{p}} \geq g_R(M) = g_{\mathfrak{q}}$.

(2) We prove the statement by induction on $g_{\mathfrak{p}}$. If $g_{\mathfrak{p}} = \infty$, then $M_{\mathfrak{p}} = 0$ and choose $\mathcal{U}$ so that $\mathfrak{p} \in \mathcal{U}$ and $M_{\mathfrak{q}} = 0$ for all $\mathfrak{q} \in \mathcal{U}$, as in Lemma 3.1(1).

If $g_{\mathfrak{p}} = 0$ then the assertion is clear.

If $0 < g_{\mathfrak{p}} < \infty$ then choose a homogeneous element $x/1 \in R_+R_{\mathfrak{p}}$ which is regular on $M_{\mathfrak{p}}$, cf. 1.3.1. By Lemma 3.1(2) we can choose an open set $\mathcal{U}_1$ such that $\mathfrak{p} \in \mathcal{U}_1$ and $x/1 \in R_+R_{\mathfrak{q}}$ is regular on $M_{\mathfrak{q}}$ or $M_{\mathfrak{q}} = 0$ for all $\mathfrak{q} \in \mathcal{U}_1$. Since $g_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/xM_{\mathfrak{p}}) = g_{\mathfrak{p}} - 1$, we can use the induction hypothesis to obtain an open set $\mathcal{U}_2$ so that $\mathfrak{p} \in \mathcal{U}_2$ and

$$g_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/xM_{\mathfrak{p}}) \leq g_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}/xM_{\mathfrak{q}}) \quad \text{for all} \quad \mathfrak{q} \in \mathcal{U}_2\,.$$

Setting $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$ we have thus for all $\mathfrak{q} \in \mathcal{U}$ with $M_{\mathfrak{q}} \neq 0$:

$$g_{\mathfrak{p}} = g_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/xM_{\mathfrak{p}}) + 1 \leq g_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}/xM_{\mathfrak{q}}) + 1 = g_{\mathfrak{q}}$$

When $M_{\mathfrak{q}} = 0$ we have $g_{\mathfrak{q}} = \infty$, hence the inequality is also satisfied. □

The next proposition can be deduced immediately from the above lemmas.

3.4. **Proposition.** *For any integer* k *the following sets are open in* $\operatorname{Spec}(R_0)$*:*

$$\mathcal{D}^k_1(M) := \{\mathfrak{q} \in \operatorname{Spec}(R_0) \mid g_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}) \geq k\}$$
$$\mathcal{D}^k_2(M) := \{\mathfrak{q} \in \operatorname{Spec}(R_0) \mid n_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}) \leq k\}$$
$$\mathcal{D}^k_3(M) := \{\mathfrak{q} \in \operatorname{Spec}(R_0) \mid n_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}) - g_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}) \leq k\}$$

Our main theorem in this section generalizes a result of Katzman and Sharp in [9, 1.8]; they prove the case $M = R$ of Theorem 3.5 below.

3.5. **Theorem.** *Let* n *be an integer such that* $H^i_{R_+}(M) = 0$ *for all* $i > n$ *and* $H^n_{R_+}(M) \neq 0$. *The set* $\operatorname{Supp}_{R_0}(H^n_{R_+}(M))$ *is then closed in* $\operatorname{Spec}(R_0)$.

In view of 1.1, this gives Theorem 1(2) in the introduction.

Proof. Using 1.2, we conclude

$$\operatorname{Supp}_{R_0}\left(H^n_{R_+}(M)\right) = \{\mathfrak{q} \in \operatorname{Spec}(R_0) \mid n_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}) = n\}$$

We also have $n_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}) \leq n$ for all $\mathfrak{q} \in \operatorname{Spec}(R_0)$, hence, in the notation of Proposition 3.4, the complement in $\operatorname{Spec}(R_0)$ of the set above is precisely the set $\mathcal{D}^{n-1}_2(M)$, which is open. □

3.6. *Remark.* Arguments similar to those in the proof above show that the support of the bottom nonvanishing local cohomology is closed, too. However the result would be weaker than what is known, since $\operatorname{Ass}_{R_0}(H^g_{R_+}(M))$ is finite for $g = \operatorname{grade}(R_+, M)$, cf. [2].

4. Cohen-Macaulay modules

In this section we consider the case when M is a Cohen-Macaulay R-module. We recall below several known facts on Cohen-Macaulay graded modules, for which we refer to [5].

4.1. Assume that $(R_0, \mathfrak{m}_0)$ is local and set $\mathfrak{m} = \mathfrak{m}_0 + R_+$, the unique graded maximal ideal of R. The R-module M is then Cohen-Macaulay if and only if the $R_\mathfrak{m}$-module $M_\mathfrak{m}$ is Cohen-Macaulay, and in this case we have:

$$\dim_R(M) = \dim_{R_\mathfrak{m}}(M_\mathfrak{m}) = \operatorname{depth}_{R_\mathfrak{m}}(M_\mathfrak{m}) = \operatorname{grade}(\mathfrak{m}, M)$$

4.2. **Proposition.** *Assume $(R_0, \mathfrak{m}_0)$ is local, and M is a nonzero Cohen-Macaulay R-module. The following then holds:*

$$\operatorname{grade}(R_+, M) = \dim_R M - \dim_{R_0} M$$

Proof. Since R is isomorphic to a quotient of a polynomial ring over R_0, we may actually assume by 1.3.4 that R is a polynomial ring $R = R_0[x_1, \dots x_s]$. (Note that dimension is also invariant under the change of the ring.) Furthermore, by 1.3.3 we may assume that R_0 is complete, and hence it is a quotient S_0/I, where S_0 is a regular local ring. It follows that R is a quotient of the polynomial ring $S_0[x_1, \dots x_s]$. Replacing R with this ring, we may thus assume that both R and R_0 are regular (in particular, Cohen-Macaulay)

In his thesis, Lim [11, 1.2.9] proved that the following holds:

$$(*) \qquad \operatorname{grade}(R_+, M) = \operatorname{ht}(R_+) + \operatorname{ht}(I \cap R_0) - \operatorname{ht} I$$

where $I = \sqrt{\operatorname{ann}_R M}$. Note that $I \cap R_0 = \sqrt{\operatorname{ann}_{R_0} M}$, using for example [16, 1.1.2(1)]. Since R_0 and R are Cohen-Macaulay, we have

$$\operatorname{ht}(I \cap R_0) = \dim R_0 - \dim_{R_0} M$$
$$\operatorname{ht}(I) = \dim R - \dim_R M$$

Since $\operatorname{ht}(R_+) = \dim(R) - \dim(R_0)$, the formula $(*)$ gives the equality in the statement. □

When R_0 is local, the notion of depth of M over R_0 can be introduced in the usual way (even if M is not necessarily finitely generated over R_0), namely as being equal to the length of a maximal regular sequence. (See [16] for more details.)

4.3. **Proposition.** *If $(R_0, \mathfrak{m}_0)$ is local and M is a Cohen-Macaulay R-module, then*

$$\dim_R(M/\mathfrak{m}_0 M) = \dim_R(M) - \operatorname{depth}_{R_0}(M)$$

Proof. Let $\mathfrak{m} = \mathfrak{m}_0 + R_+$ be the unique graded maximal ideal of R. Consider a maximal M-regular sequence $q_1, \dots q_s$ in $\mathfrak{m}_0$, and choose $t_1, \dots t_r \in \mathfrak{m}$ such that $q_1, \dots q_s, t_1, \dots, t_r$ is a maximal M-regular sequence in $\mathfrak{m}$. We have thus:

$$\operatorname{depth}_{R_0}(M) = s \quad \text{and} \quad \operatorname{grade}(\mathfrak{m}, M) = r + s$$

Using 4.1 we evaluate the right-hand part of the equality in the statement:

$$\dim_R(M) - \operatorname{depth}_{R_0}(M) = \operatorname{grade}(\mathfrak{m}, M) - \operatorname{depth}_{R_0}(M) = r$$

Set $\overline{M} = M/(q_1, \dots q_s)M$. Note that $\overline{M}$ is a graded Cohen-Macaulay R-module and in view of 4.1 we have

$$\dim_R(\overline{M}) = \operatorname{grade}(\mathfrak{m}, \overline{M}) = r$$

To prove the statement, it suffices thus to show that the R-modules $M/\mathfrak{m}_0M$ and $\overline{M}$ have the same dimension. Since $M/\mathfrak{m}_0M$ is a homomorphic image of $\overline{M}$, we have $\dim_R(M/\mathfrak{m}_0M) \le \dim_R(\overline{M})$.

To prove the reverse inequality, consider $Q \in \operatorname{Ass}_R(\overline{M})$ such that $Q \cap R_0 = \mathfrak{m}_0$. We can choose such a prime because $\operatorname{depth}_{R_0}(\overline{M}) = 0$, hence $\mathfrak{m}_0 \in \operatorname{Ass}_{R_0}(\overline{M})$, and we can apply for example [16, 2.1.2].

The R-module $\overline{M}$ is graded Cohen-Macaulay, hence the $R_\mathfrak{m}$-module $\overline{M}_\mathfrak{m}$ is Cohen-Macaulay. Since $Q_\mathfrak{m}$ is an associated prime of this last module, we have

$$\dim_R(\overline{M}) = \dim_{R_\mathfrak{m}}(\overline{M}_\mathfrak{m}) = \dim_{R_\mathfrak{m}}(R_\mathfrak{m}/Q_\mathfrak{m}) = \dim_R(R/Q) \tag{4.3.1}$$

Let $\overline{N}$ be a Q-primary submodule $\overline{N} \subseteq \overline{M}$, and let N denote the preimage of $\overline{N}$ in M. We have $\operatorname{Ass}_R(M/N) = \operatorname{Ass}_R(\overline{M}/\overline{N}) = \{Q\}$, hence:

$$\dim_R(M/N) = \dim_R(R/Q) \tag{4.3.2}$$

It also follows that $\operatorname{rad}(\operatorname{ann}_R(M/N)) = Q$, and since $\mathfrak{m}_0 \subseteq Q$, we conclude that there exists an integer r such that $\mathfrak{m}_0^r \subseteq \operatorname{ann}_R(M/N)$, and hence $\mathfrak{m}_0^r M \subseteq N$. We have thus:

$$\dim_R(M/\mathfrak{m}_0^r M) \ge \dim_R(M/N) \tag{4.3.3}$$

On the other hand, by [16, 5.1] we have:

$$\dim_R(M/\mathfrak{m}_0^r M) = \dim_R(M/\mathfrak{m}_0 M) \tag{4.3.4}$$

Putting together the four equations displayed above, we obtain

$$\dim_R(M/\mathfrak{m}_0M) \ge \dim_R(\overline{M})$$

and this finishes the proof. □

4.4. When R_0 is local, we set

$$\operatorname{codepth}_{R_0}(M) = \dim_{R_0}(M) - \operatorname{depth}_{R_0}(M)$$

Recall from [16, 1.2.2] that we have the following formulas:

$$\begin{aligned} \operatorname{depth}_{R_0}(M) &= \inf\{\operatorname{depth}_{R_0}(M_i) \mid i \in \mathbb{Z} \quad \text{with} \quad M_i \neq 0\} \\ \dim_{R_0}(M) &= \sup\{\dim_{R_0}(M_i) \mid i \in \mathbb{Z}\} \end{aligned}$$

Note that if $M \neq 0$ then $0 \le \operatorname{codepth}_{R_0}(M) \le \dim(R_0)$ and if $M = 0$ then $\operatorname{codepth}_{R_0}(M) = -\infty$.

Using the notation introduced earlier, Propositions 4.2 and 4.3 prove that the following formula holds whenever R_0 is local, and M is a nonzero Cohen-Macaulay R-module:

$$n_R(M) - g_R(M) = \operatorname{codepth}_{R_0}(M) \tag{4.4.1}$$

Note that the formula also holds when $M = 0$.

4.5. *Remark.* The R_0-module M is Cohen-Macaulay (meaning that $\operatorname{codepth}_{R_0} M \le 0$) if and only if for each i the R_0-module M_i is Cohen-Macaulay and $\dim_{R_0} M_i = \dim_{R_0} M$.

Indeed, if M is Cohen-Macaulay, then, using 4.4, we have for all i:

$$\operatorname{depth}_{R_0} M_i \ge \operatorname{depth}_{R_0} M = \dim_{R_0} M \ge \dim_{R_0} M_i$$

It follows that equalities hold above, and in particular M_i is Cohen-Macaulay.

Conversely, choose i be such that $\operatorname{depth}_{R_0} M = \operatorname{depth}_{R_0} M_i$, cf. 4.4. Since $\operatorname{depth}_{R_0} M_i = \dim_{R_0} M_i = \dim_{R_0} M$, it follows that M is Cohen-Macaulay.

In view of 1.2, we can use the formula (4.4.1) and Remark 4.5 to give a necessary and sufficient condition for a Cohen-Macaulay R-module to have only one nonvanishing local cohomology.

4.6. **Corollary.** *Assume that M is a Cohen-Macaulay R-module and R_0 is local. The following statements are then equivalent:*

(1) *M is Cohen-Macaulay as an R_0-module.*
(2) *The R_0-module M_i is Cohen-Macaulay, with $\dim_{R_0} M_i = \dim_{R_0} M$ for all i.*
(3) *There exists an integer $j \geq 0$ such that $H^i_{R_+}(M) = 0$ for all $i \neq j$.* □

In the remaining of the section we eliminate the condition that R_0 is local. Note that the formula (4.4.1) gives:

4.7. **Corollary.** *Assume that M is a Cohen-Macaulay R-module. For any $\mathfrak{p} \in \operatorname{Spec}(R_0)$ the following equality holds:*

$$n_{R_\mathfrak{p}}(M_\mathfrak{p}) - g_{R_\mathfrak{p}}(M_\mathfrak{p}) = \operatorname{codepth}_{(R_0)_\mathfrak{p}}(M_\mathfrak{p}) \qquad \square$$

Combining Corollary 4.7 with Proposition 3.4, we obtain:

4.8. **Corollary.** *If M is Cohen-Macaulay over R, then for any $k \in \mathbb{N}$ the set*

$$U^0_{C_n}(M) = \{\mathfrak{p} \in \operatorname{Spec}(R_0) \mid \operatorname{codepth}_{(R_0)_\mathfrak{p}}(M_\mathfrak{p}) \leq k\}$$

is open in $\operatorname{Spec}(R_0)$. □

Furthermore, by [16, 3.3] we have:

4.9. **Corollary.** *If M is Cohen-Macaulay over R, then for any $k \in \mathbb{N}$ the set*

$$U^0_{S_n}(M) = \{\mathfrak{p} \in \operatorname{Spec}(R_0) \mid \textit{the } (R_0)_\mathfrak{p}\textit{-module } M_\mathfrak{p} \textit{ satisfies } (S_k)\}$$

is open in $\operatorname{Spec}(R_0)$. □

5. BASE RINGS OF SMALL DIMENSION

In this section we prove in several cases that the support of the local cohomology is closed. To prove that a set is open, we use the *topological Nagata criterion*, cf. [14, 24.2], as recalled below:

5.1. A set $\mathcal{D}$ in $\operatorname{Spec}(R_0)$ is open if and only if the following two conditions are satisfied:

(1) If $\mathfrak{q} \in \mathcal{D}$ and $\mathfrak{p} \subseteq \mathfrak{q}$, then $\mathfrak{p} \in \mathcal{D}$.
(2) For any prime $\mathfrak{p} \in \mathcal{D}$ there exists an open non-empty subset of $V(\mathfrak{p})$ contained in $\mathcal{D}$, that is, there exists an open set $\mathcal{U}$ in $\operatorname{Spec}(R_0)$ such that $\emptyset \neq \mathcal{U} \cap V(\mathfrak{p}) \subseteq \mathcal{D}$.

The assumptions on R and M are as in Section 1. To simplify the notation, for every $\mathfrak{p} \in \operatorname{Spec}(R_0)$ we set:

$$g_\mathfrak{p} = g_{R_\mathfrak{p}}(M_\mathfrak{p}) \quad \text{and} \quad n_\mathfrak{p} = n_{R_\mathfrak{p}}(M_\mathfrak{p})$$

5.2. **Proposition.** *Assume that M is a Cohen-Macaulay R-module and the inequality* $\operatorname{codepth}_{(R_0)_\mathfrak{p}}(M_\mathfrak{p}) \leq 1$ *holds for all* $\mathfrak{p} \in \operatorname{Spec}(R_0)$.

The set $\operatorname{Supp}_{R_0}(H^i_{R_+}(M))$ *is then closed in* $\operatorname{Spec}(R_0)$ *for all i.*

Proof. Fix some i. We will prove the conditions (1) and (2) of 5.1 for the set $\mathcal{D} = \operatorname{Spec}(R_0) \smallsetminus \operatorname{Supp}_{R_0}(H^i_{R_+}(M))$. By Flat Base Change, we have

$$\mathcal{D} = \{\mathfrak{p} \in \operatorname{Spec}(R_0) \mid H^i_{R_+R_\mathfrak{p}}(M_\mathfrak{p}) = 0\}$$

and we see that (1) is satisfied. To check (2), let $\mathfrak{p} \in \mathcal{D}$. If $M_\mathfrak{p}$=0, then we choose $\mathcal{U}$ as in Lemma 3.1(1).

If $M_\mathfrak{p} \neq 0$, then Corollary 4.7 gives $g_\mathfrak{p} \leq n_\mathfrak{p} \leq g_\mathfrak{p} + 1$. Since $H^i_{R_+R_\mathfrak{p}}(M_\mathfrak{p}) = 0$, we have $i \notin \{g_\mathfrak{p}, n_\mathfrak{p}\}$ by 1.2. Using Theorems 3.2 and 3.3 we can choose an open set $\mathcal{U}$ such that $\mathfrak{p} \in \mathcal{U}$ and $n_\mathfrak{q} = n_\mathfrak{p}$ and $g_\mathfrak{q} = g_\mathfrak{p}$ for all $\mathfrak{q} \in \mathcal{U} \cap V(\mathfrak{p})$. Thus, for all $\mathfrak{q} \in \mathcal{U} \cap V(\mathfrak{p})$ we have $g_\mathfrak{q} \leq n_\mathfrak{q} \leq g_\mathfrak{q} + 1$ and $i \notin \{g_\mathfrak{q}, n_\mathfrak{q}\}$. Using again 1.2 it follows that $H^i_{R_+R_\mathfrak{q}}(M_\mathfrak{q}) = 0$, that is, $\mathfrak{q} \in \mathcal{D}$. □

5.3. **Theorem.** *Assume that M is a Cohen-Macaulay R-module.*
If $\dim R_0 \leq 2$, then $\operatorname{Supp}_{R_0}(H^i_{R_+}(M))$ is closed for all i.

Proof. Fix some i. As above, we only need to check condition (2) of 5.1 for the set $\mathcal{D} = \operatorname{Spec}(R_0) \smallsetminus \operatorname{Supp}_{R_0}(H^i_{R_+}(M))$. Let $\mathfrak{p} \in \mathcal{D}$. We distinguish the following two cases:

(a) $\operatorname{ht}\mathfrak{p} = 2$. In this case we take $\mathcal{U} = \operatorname{Spec}(R_0)$, noting that $\mathcal{U} \cap V(\mathfrak{p}) = \{\mathfrak{p}\}$.

(b) $\operatorname{ht}\mathfrak{p} \leq 1$. In this case we have $\operatorname{codepth}_{(R_0)_\mathfrak{p}}(M_\mathfrak{p}) \leq \dim(R_0)_\mathfrak{p} \leq 1$, hence $g_\mathfrak{p} \leq n_\mathfrak{p} \leq g_\mathfrak{p} + 1$ when $M_\mathfrak{p} \neq 0$, by Corollary 4.7. We proceed as in the proof of Proposition 5.2. □

5.4. **Theorem.** *Assume that R_0 is semilocal, and M is a Cohen-Macaulay R-module. If $\dim R_0 \leq 3$, then $\operatorname{Supp}_{R_0}(H^i_{R_+}(M))$ is closed for all i.*

Proof. Fix some i. As above, we only need to check condition (2) of 5.1 for the set $\mathcal{D} = \operatorname{Spec}(R_0) \smallsetminus \operatorname{Supp}_{R_0}(H^i_{R_+}(M))$. Let $\mathfrak{p} \in \mathcal{D}$. We distinguish the following three cases:

(a) If $\operatorname{ht}\mathfrak{p} = 3$, then we take $\mathcal{U} = \operatorname{Spec}(R_0)$.

(b) If $\operatorname{ht}\mathfrak{p} = 2$, then we take $\mathcal{U} = \operatorname{Spec}(R_0) \smallsetminus \{\mathfrak{m}_1, \ldots, \mathfrak{m}_s\}$, where $\mathfrak{m}_1, \ldots, \mathfrak{m}_s$ are the maximal ideals of R_0.

(c) If $\operatorname{ht}\mathfrak{p} \leq 1$, then by Corollary 4.7 we have $g_\mathfrak{p} \leq n_\mathfrak{p} \leq g_\mathfrak{p} + 1$ when $M_\mathfrak{p} \neq 0$ and we proceed as in the proof of Proposition 5.2. □

We recall a result of Brodmann, Fumasoli and Lim [1]:

5.5. If R_0 is semilocal of dimension at most 2, then $H^i_{R_+}(M)$ is tame for all i.

Lim [13] has proved tameness of the local cohomology for any ring of dimension at most 2, under the additional assumption that M is Cohen-Macaulay. Using our methods, we recover below Lim's result.

5.6. **Theorem.** *Assume that M is a Cohen-Macaulay R-module.*
If $\dim R_0 \leq 2$, then $H^i_{R_+}(M)$ is tame for all i.

Proof. Fix some i. If there exists some $\mathfrak{p} \in \operatorname{Spec}(R_0)$ such that $H^i_{R_+R_\mathfrak{p}}(M_\mathfrak{p})_d \neq 0$ for infinitely many $d < 0$, then 5.5 yields that $H^i_{R_+R_\mathfrak{p}}(M_\mathfrak{p})_j \neq 0$ for all $j \ll 0$, hence $H^i_{R_+}(M)_j \neq 0$ for all $j \ll 0$, and thus $H^i_{R_+}(M)$ is tame.

It remains thus to study the case when $H^i_{R_+R_\mathfrak{p}}(M_\mathfrak{p})$ is a finite $(R_0)_\mathfrak{p}$-module for all $\mathfrak{p} \in \operatorname{Spec}(R_0)$. Note that if $i = 0$, then $H^0_{R_+}(M)$ is itself a finite R_0-module.

Also, if $i = 1$, then it follows that $H^1_{R_+}(M)$ is a finite R_0-module as well, using a result of Faltings, cf. [4, 9.6.1]. Thus, we may also assume $i > 1$.

Claim: For every integer n the following set is open:

$$\mathcal{D}_n = \{\mathfrak{p} \in \operatorname{Spec}(R_0) \mid H^i_{R_+R_\mathfrak{p}}(M_\mathfrak{p})_j = 0 \quad \text{for all} \quad j \le -n\}$$

Assuming the claim, we will prove the theorem. Set $Z_n = \operatorname{Spec}(R_0) \smallsetminus \mathcal{D}_n$.

If $Z_n = \emptyset$ for some n, then $H^i_{R_+R_\mathfrak{p}}(M_\mathfrak{p})_j = 0$ for all $j \le -n$ and all $\mathfrak{p} \in \operatorname{Spec}(R_0)$, hence $H^i_{R_+}(M)_j = 0$ for all $j \le -n$, and thus $H^i_{R_+}(M)$ is tame.

Assume now that for each n we have $Z_n \neq \emptyset$. The following chain of closed subsets:

$$\cdots \subseteq Z_n \subseteq Z_{n-1} \subseteq \cdots \subseteq Z_0$$

shows that there exists an n_0 such that $Z_n = Z_{n_0}$ for all $n \ge n_0$. If $\mathfrak{p} \in Z_{n_0}$, then $\mathfrak{p} \in Z_n$ for all $n \ge n_0$, and it follows that $H^i_{R_+R_\mathfrak{p}}(M_\mathfrak{p})$ is nonzero in infinitely many degrees; this case was treated in the beginning.

Proof of the claim: To prove that the set $\mathcal{D}_n$ is open we will use again the criterion 5.1. We only need to prove (2), that is: for each $\mathfrak{p} \in \mathcal{D}_n$ we need to find an open set $\mathcal{U}$ of $\operatorname{Spec}(R_0)$ such that $\emptyset \neq \mathcal{U} \cap V(\mathfrak{p}) \subseteq \mathcal{D}_n$. We have the following two cases:

(a) If $\operatorname{ht}\mathfrak{p} = 2$, then take $\mathcal{U} = \operatorname{Spec}(R_0)$, and note that $\mathcal{U} \cap V(\mathfrak{p}) = \{\mathfrak{p}\}$.

(b) Assume now $\operatorname{ht}\mathfrak{p} \le 1$. If $M_\mathfrak{p} = 0$, then choose $\mathcal{U}$ as in Lemma 3.1(1). We may assume thus $M_\mathfrak{p} \neq 0$. Since $\operatorname{codepth}_{(R_0)_\mathfrak{p}}(M_\mathfrak{p}) \le 1$, Corollary 4.7 implies

$$g_\mathfrak{p} \le n_\mathfrak{p} \le g_\mathfrak{p} + 1 \tag{5.6.1}$$

If $g_\mathfrak{p} = 0$, then $n_\mathfrak{p} \le 1$. Let $\mathcal{U}$ be an open set containing $\mathfrak{p}$ such that $n_\mathfrak{q} \le n_\mathfrak{p}$, and thus $n_\mathfrak{q} \le 1$, for all $\mathfrak{q} \in \mathcal{U}$. Since $i > 1$, we have then $H^i_{R_+R_\mathfrak{q}}(M_\mathfrak{q}) = 0$ for all $\mathfrak{q} \in \mathcal{U}$, hence $\mathcal{U} \subseteq \mathcal{D}_n$.

Assume now that $g_\mathfrak{p} > 0$. Note that $i \neq n_\mathfrak{p}$. Indeed, if $i = n_\mathfrak{p}$, then $H^i_{R_+R_\mathfrak{p}}(M_\mathfrak{p})$, and thus $H^i_{R_+}(M)$, is nonzero in infinitely many degrees by Remark 2.6. This contradicts $\mathfrak{p} \in \mathcal{D}_n$.

If $i \neq g_\mathfrak{p}$ then it suffices to choose an open set $\mathcal{U}$ such that $\mathfrak{p} \in \mathcal{U}$ and $g_\mathfrak{p} = g_\mathfrak{q}$ and $n_\mathfrak{p} = n_\mathfrak{q}$ for all $\mathfrak{q} \in \mathcal{U} \cap V(\mathfrak{p})$; this can be done using Lemmas 3.2 and 3.3. For all such $\mathfrak{q}$ we have $i \neq n_\mathfrak{q}$ and $i \neq g_\mathfrak{q}$. In view of (5.6.1) and 1.2 we conclude $H^i_{R_+R_\mathfrak{q}}(M_\mathfrak{q}) = 0$, and thus $\mathfrak{q} \in \mathcal{D}_n$.

If $i = g_\mathfrak{p}$, consider an open set $\mathcal{U}_1 = \{\mathfrak{q} \in \operatorname{Spec}(R_0) \mid b \notin \mathfrak{q}\}$ with $b \in R_0$ such that $\mathfrak{p} \in \mathcal{U}_1$ and $g_\mathfrak{q} \ge g_\mathfrak{p}$ for all $\mathfrak{q} \in \mathcal{U}_1$ (by Lemma 3.3). The primes of the ring $(R_0)_b$ correspond bijectively with the elements of $\mathcal{U}_1$. By assumption $H^i_{R_+R_\mathfrak{q}}(M_\mathfrak{q})$ is finite over R_0 for all $\mathfrak{q} \in \mathcal{U}_1$. Since $g_\mathfrak{q} \ge i$ for all such $\mathfrak{q}$, it actually follows that $H^j_{R_+R_\mathfrak{q}}(M_\mathfrak{q})$ is finitely generated for all $j < i+1$. By a theorem a Faltings [4, 9.6.1], it follows that $H^j_{R_+R_b}(M_b)$ is finitely generated for all $j < i+1$, and hence there exists an integer d such that $H^i_{R_+R_b}(M_b)_l = 0$ for all $l \le d$. In particular, it follows that $H^i_{R_+R_\mathfrak{q}}(M_\mathfrak{q})_l = 0$ for all $\mathfrak{q} \in \mathcal{U}_1$ and all $l \le d$. If $d \ge -n$ then we take $\mathcal{U} = \mathcal{U}_1$. If $d < -n$ then we take $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$ where

$$\mathcal{U}_2 = \bigcap_{d<j\le -n} \operatorname{Spec}(R_0) \smallsetminus \operatorname{Supp}_{R_0}(H^i_{R_+}(M)_j)$$

□

Note. After the submission of this paper, we learned that the results of Section 2 have been obtained independently by Sazeedeh. Also, M. Brodmann informed us that he has a different proof of Theorem 3.5.

References

[1] M. P. Brodmann, S. Fumasoli, C. S. Lim, *Low-codimensional associated primes of graded components of local cohomology modules* , J. Algebra, to appear.
[2] M. P. Brodmann, F. A. Lashgari, *A finiteness result for associated primes of local cohomology modules*, Proc. Amer. Math. Soc. **128** (2000) 2851–2853.
[3] M. P. Brodmann, M. Hellus, *Cohomological patterns of coherent sheaves over projective schemes*, J. Pure Appl. Algebra **172** (2002) 165–182.
[4] M. P. Brodmann, R. Y. Sharp, *Local cohomology: an introduction with geometric applications*, Cambridge University Press, 1998.
[5] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies Adv. Math. **39**, Unversity Press, Cambridge, 1993.
[6] A. Grothendieck, *Éléments de géométrie algébrique IV*, Inst. Hautes Études Sci. Publ. Math. **24**, 1965.
[7] C. Huneke, *Problems on local cohomology*, Free resolutions in commutative algebra and algebraic geometry, Sundance '90, Res. Notes in Math **2**, Jones and Barlett, 1992, pp. 93-108.
[8] M. Katzman, *The support of top graded local cohomology modules*, Commutative Algebra with a focus on geometric and homological aspects, Lecture Notes in Pure and Appl. Math, Marcel Dekker, to appear.
[9] M. Katzman, R. Y. Sharp, *Some properties of the top graded local cohomology modules*, J. Algebra **259** (2003) 599–612.
[10] D. Kirby, *Artinian modules and Hilbert polynomials*, Quart. J. Math. Oxford Ser. (2) **24** (1973) 45–47.
[11] C. S. Lim, *Graded local cohomology and its associated primes*, Ph. D. Thesis, Michigan State University, 2002.
[12] C. S. Lim, *Graded local cohomology modules and their associated primes: the Cohen-Macaulay case*, J. Pure Appl. Algebra **185** (2003) 225-238.
[13] C. S. Lim, preprint (2004).
[14] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced mathematics, Vol 8, Cambridge, 1986.
[15] L. Melkersson, *On asymptotic stability for sets of prime ideals connected with the powers of an ideal*, Math. Proc. Cambridge Philos. Soc. **107** (1990) 267–271.
[16] C. Rotthaus, L. M. Şega, *Open loci of graded modules*, preprint (2004), arXiv: math.AC/0403399.
[17] A. K. Singh, *p-torsion elements in local cohomology*, Math. Res. Letters **7** (2000) 165–176.

Department of Mathematics, Michigan State University, East Lansing, MI 48824
E-mail address: `rotthaus@math.msu.edu`

Department of Mathematics, Michigan State University, East Lansing, MI 48824
E-mail address: `lsega@math.msu.edu`