HOMOLOGICAL PROPERTIES OF POWERS OF THE MAXIMAL IDEAL OF A LOCAL RING

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Abstract. It is known that the powers $\mathfrak{m}^n$ of the maximal ideal of a local Noetherian ring share certain homological properties for all sufficiently large integers n. For example, the natural homomorphisms $R \to R/\mathfrak{m}^n$ are Golod, respectively, small, for all large n. We give effective bounds on the smallest integers n for which such properties begin to hold.

Introduction

Let R be a local commutative Noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. We study homological properties of the powers $\mathfrak{m}^n$ which hold for all large values of n.

One such property is established by Levin [19]; he proves that the natural homomorphism $R \to R/\mathfrak{m}^n$ is Golod for all large n. Lee [17] defines the Golod invariant of R to be the smallest number s such that $R \to R/\mathfrak{m}^n$ is Golod for all $n \geq s$. The results of this paper are better stated in terms of a Golod index $\boldsymbol{G}(R)$, defined to be one less than the invariant introduced by Lee.

In order to study the Golod property, we consider two related homological properties. One is based on the notion of small homomorphism introduced by Avramov [3], the other arises from Levin's proof of his theorem; we refer to the corresponding sections for precise definitions. We define indices $\boldsymbol{A}(R)$ and $\boldsymbol{L}(R)$, in analogy to the Golod index. Results of Avramov and Levin show that these are natural numbers that provide bounds for $\boldsymbol{G}(R)$ as follows:

$$\boldsymbol{A}(R) \leq \boldsymbol{G}(R) \leq \boldsymbol{L}(R)\,.$$

We obtain bounds for $\boldsymbol{A}(R)$ and $\boldsymbol{L}(R)$ in terms of numerical invariants of the associated graded ring $\operatorname{gr}_{\mathfrak{m}}(R)$ with respect to the $\mathfrak{m}$-adic filtration. Recall that the $\mathfrak{m}$-adic completion $\widehat{R}$ has a *minimal Cohen presentation* $\widehat{R} \cong Q/\mathfrak{a}$, with $(Q, \mathfrak{n})$ a regular local ring and $\mathfrak{a} \subseteq \mathfrak{n}^2$. We summarize below our results:

Theorem 1. *Let $(R, \mathfrak{m})$ be a local Noetherian ring with $\mathfrak{m} \neq 0$. Let $\widehat{R} \cong Q/\mathfrak{a}$ be a minimal Cohen presentation and let $\operatorname{pol\,reg}(R)$ denote the Castelnuovo-Mumford regularity of $\operatorname{gr}_{\mathfrak{m}}(R)$ over $\operatorname{gr}_{\mathfrak{n}}(Q)$. The following then hold:*

(1) $\inf\{i \geq 1 \mid \mathfrak{a} \cap \mathfrak{n}^{i+2} \subseteq \mathfrak{n}\mathfrak{a}\} \leq \boldsymbol{A}(R) \leq \boldsymbol{L}(R) \leq \max\{1, \operatorname{pol\,reg}(R)\}$.
(2) *If R is a complete intersection, then* $\boldsymbol{A}(R) = \inf\{i \geq 1 \mid \mathfrak{a} \cap \mathfrak{n}^{i+2} \subseteq \mathfrak{n}\mathfrak{a}\}$.
(3) *If R is a hypersurface or a Golod Artinian ring, then*

$$\boldsymbol{A}(R) = \boldsymbol{G}(R) = \boldsymbol{L}(R) = \max\{1, \operatorname{pol\,reg}(R)\}\,.$$

(4) *If* $\operatorname{edim} R - \dim R \leq 1$, *or if* $\operatorname{edim} R \leq 2$, *or if* $\operatorname{edim} R = 3$ *and R is a complete intersection, then* $\boldsymbol{A}(R) = \boldsymbol{G}(R)$.
(5) *The graded k-algebra $\operatorname{gr}_{\mathfrak{m}}(R)$ is Koszul if and only if* $\boldsymbol{L}(R) = 1$.

We study the behavior of the indices under factorization of regular sequences. Part (2) of the next theorem answers partially a question of Roos [29].

Theorem 2. *For each local Noetherian ring $(R, \mathfrak{m})$ the following hold:*

(1) *If x is a regular element such that $(x) \neq \mathfrak{m}$, then $\boldsymbol{A}(R) \leq \boldsymbol{A}(R/(x))$.*
(2) *If $x \notin \mathfrak{m}^2$ and the initial form of x is $\operatorname{gr}_{\mathfrak{m}}(R)$-regular, then $\boldsymbol{L}(R) = \boldsymbol{L}(R/(\boldsymbol{x}))$.*

The index $\boldsymbol{L}(R)$ is a particular case of an index defined in Section 3 for any R-module M, and denoted $\boldsymbol{L}_R(M)$. If $\operatorname{pol reg}(M)$ denotes the Castelnuovo-Mumford regularity of $\operatorname{gr}_{\mathfrak{m}}(M)$ over a certain polynomial ring, then we obtain an inequality $\boldsymbol{L}_R(M) \leq \operatorname{pol reg}(M) + 1$, which gives the corresponding inequality of Theorem 1(1). We obtain an application to delta invariants $\delta^i_R(M)$, defined by Auslander [2] when R is Gorenstein, and by Martsinkovsky [24] in general.

Theorem 3. *If M is a finite module over a non-regular local ring $(R, \mathfrak{m}, k)$, then $\delta^i_R(\mathfrak{m}^n M) = 0$ for all $i \geq 0$ and all $n > \operatorname{pol reg}(M)$.*

This generalizes a theorem of Yoshino [39], which shows that if R is Gorenstein, then $\delta^i_R(\mathfrak{m}^n) = 0$ for all large n.

In Section 1 we discuss notions of regularity; the definitions involve Castelnuovo-Mumford regularity over different graded rings of the associated graded module $\operatorname{gr}_{\mathfrak{m}}(M)$. In Section 2 we construct canonical homomorphisms

$$\tau_i^M \colon \operatorname{Tor}_i^R(M, k) \to \operatorname{Tor}_i^{\operatorname{gr}_{\mathfrak{m}}(R)}\left(\operatorname{gr}_{\mathfrak{m}}(M), k\right),$$

which can be computed by means of free resolutions of either the first or the second module argument. We show that M has a linear resolution, that is, $\operatorname{reg}_R(M) = 0$, if and only if the maps τ_i^M are bijective for all i.

In Section 3 we introduce for each finite R-module M the index $\boldsymbol{L}_R(M)$ and we bound it by $\operatorname{pol reg}(M)+1$. As consequences, we obtain effective versions of results of Levin and Avramov on Poincaré series, and of similar results of Roos for Bass series. Theorem 3 is proved in Section 4. In Section 5 we define the index $\boldsymbol{A}(R)$ and use results of Avramov [3] to establish the relevant parts of Theorems 1 and 2. In Section 6 we discuss the invariant $\boldsymbol{G}(R)$ and prove Theorem 1(4). We also prove that if $\operatorname{edim} R - \dim R \leq 1$, then the ring $R/\mathfrak{m}^n$ is Golod for all integers $n \geq 2$. Since the equality $\operatorname{edim} R - \dim R = 0$ characterizes regularity, this generalizes Golod's classical example [12].

In Section 7 we introduce the index $\boldsymbol{L}(R)$ by the formula $\boldsymbol{L}(R) = \boldsymbol{L}_R(\mathfrak{m})$ and derive the equalities of Theorem 1(3) from results proved earlier in the paper. We also prove Theorem 1(5), which extends a characterization of graded Koszul algebras noted by Roos [29]. In particular, this shows that if R is Koszul, that is, $\operatorname{reg}_R(k) = 0$, then $\operatorname{Ext}^*_R(k, k)$ is finitely generated as a graded algebra under Yoneda product. While these two conditions are equivalent for graded algebras, we give an example of a local ring for which they are not.

In Section 8 we prove Theorem 2(2). In Section 9 we consider graded rings. Adapting our definitions to this case, we prove that if R is a graded Golod ring which is not a field, then $\boldsymbol{A}(R) = \boldsymbol{G}(R) = \boldsymbol{L}(R) = \max\{1, \operatorname{pol reg}(R)\}$. We end the paper with various examples.

1. Regularity of modules

In this paper all rings are commutative Noetherian and all modules are assumed finitely generated.

1.1. If A is a ring and $\boldsymbol{a} = a_1, \dots, a_e$ is a sequence of elements of A, then $K(\boldsymbol{a}; A)$ denotes the Koszul complex on $\boldsymbol{a}$. For an A-module N we set $K(\boldsymbol{a}; N) = K(\boldsymbol{a}; A) \otimes_A N$ and $\mathrm{H}_*(\boldsymbol{a}; N) = \mathrm{H}_*\big(K(\boldsymbol{a}; N)\big)$. If $\varphi\colon A \to B$ is a ring homomorphism, then clearly $K(\boldsymbol{a}; A) \otimes_A B = K(\varphi(\boldsymbol{a}); B)$. If the A-module structure of N is induced through the homomorphism φ, then we systematically identify $K(\boldsymbol{a}; N)$ and $K(\varphi(\boldsymbol{a}); N)$.

1.2. Let k be a field, $A = \bigoplus_{n=0}^{\infty} A_n$ a commutative graded algebra with $A_0 = k$, and $N = \bigoplus_{n=0}^{\infty} N_n$ a graded A-module. For each $d \in \mathbb{Z}$ we denote $N(d)$ the graded A-module with $N(d)_p = N_{d+p}$. We denote A_+ the maximal homogeneous ideal of A and write k for the residue field $A/A_{\geqslant 1}$ modulo the maximal homogeneous ideal of A. The module N has a minimal graded free resolution

$$G = \quad \cdots \to G_i \xrightarrow{\partial_i} G_{i-1} \to \cdots \to G_1 \xrightarrow{\partial_1} G_0\,,$$

where for each i the module G_i is isomorphic to a direct sum of copies of $A(-j)$ and $\partial_i(G_i) \subseteq (A_{\geqslant 1})G_{i-1}$. Any two minimal graded free resolutions are isomorphic as complexes of graded A-modules, so the number of direct summands of G_i isomorphic to $A(-j)$ is an invariant of N, called the ij'th *graded Betti number* $\beta^A_{ij}(N)$. The *Castelnuovo-Mumford regularity* of N is the number

$$\operatorname{reg}_A(N) = \sup\{s \in \mathbb{Z} \mid \beta^A_{i,i+s}(N) \neq 0 \quad \text{for some} \quad i \in \mathbb{N}\}\,.$$

We note that $\beta^A_{i,j}(N) = \operatorname{rank}_k \operatorname{Tor}^A_i(N,k)_j$; these numbers can be calculated from any free resolution of k or N over A.

1.3. Assume further that $A = A_0[A_1]$. We can present A as $k[\boldsymbol{u}]/I$, where $k[\boldsymbol{u}]$ is a polynomial ring over k with variables $\boldsymbol{u} = u_1, \dots, u_r$ in degree 1 and I a homogeneous ideal. We define the *polynomial regularity* of N by the formula

$$\operatorname{pol\,reg}(N) = \operatorname{reg}_{k[\boldsymbol{u}]}(N)\,.$$

The next lemma shows that the right-hand side does not depend on the choice of the presentation.

1.4. **Lemma.** *If $A = k[\boldsymbol{u}]/I$ and $A = k[\boldsymbol{v}]/J$ are two presentations as above, then $\operatorname{reg}_{k[\boldsymbol{u}]}(N) = \operatorname{reg}_{k[\boldsymbol{v}]}(N)$.*

Proof. The canonical maps $k[\boldsymbol{u}] \xrightarrow{\alpha} A \xleftarrow{\beta} k[\boldsymbol{v}]$ define a surjective homomorphism

$$k[\boldsymbol{u}, \boldsymbol{v}] = k[\boldsymbol{u}] \otimes_k k[\boldsymbol{v}] \to A$$

through which both α and β factor, so it suffices to prove that $\operatorname{reg}_{k[\boldsymbol{u}]}(N) = \operatorname{reg}_{k[\boldsymbol{u},\boldsymbol{v}]}(N)$. Using induction on the number of variables in $\boldsymbol{v}$, it suffices to show that $\operatorname{reg}_P(N) = \operatorname{reg}_{P[v]}(N)$, where $P = k[\boldsymbol{u}]$ and v is a single variable which acts on N through a surjective homomorphism $\gamma\colon P[v] \to A$ such that $\gamma|_P = \alpha$. Replacing v with the linear form $v - \sum_{j=1}^r a_j u_j$, where $a_j \in k$ are such that $\gamma(v) = \alpha(\sum_{j=1}^r a_j u_j)$, we can assume $vN = 0$. By the graded graded version of a well-known result (see [10, 1.6.13]) we have an exact sequence

$$\cdots \to \mathrm{H}_i(\boldsymbol{u}; N) \to \mathrm{H}_i(\boldsymbol{u}, v; N) \to \mathrm{H}_{i-1}(\boldsymbol{u}; N(-1)) \xrightarrow{\pm v} \mathrm{H}_{i-1}(\boldsymbol{u}; N) \to \cdots$$

of Koszul homology. Since $vN = 0$, it splits into short exact sequences

$$0 \to \mathrm{H}_i(\boldsymbol{u}; N) \to \mathrm{H}_i(\boldsymbol{u}, v; N) \to \mathrm{H}_{i-1}(\boldsymbol{u}; N(-1)) \to 0\,.$$

The complex $K(\boldsymbol{u};P)$, respectively $K(\boldsymbol{u},v;P[v])$, is a minimal graded free resolution of k over P, respectively $P[v]$, hence we have

$$\beta^P_{i,j}(N)=\operatorname{rank}_k\big(\mathrm{H}_i(\boldsymbol{u};N)\big)_j \quad\text{and}\quad \beta^{P[v]}_{i,j}(N)=\operatorname{rank}_k\big(\mathrm{H}_i(\boldsymbol{u},v;N)\big)_j\,.$$

Computing ranks from the exact sequences above, we obtain

$$\beta^P_{i,j}(N)+\beta^P_{i-1,j-1}(N)=\beta^{P[v]}_{i,j}(N)$$

for all $i,j\in\mathbb{Z}$. These equalities show that the regularities coincide. □

1.5. If a graded A-module N has finite length, then $\operatorname{pol\,reg}(N)$ can be expressed as the largest integer s for which $N_s\neq 0$. Indeed, let $A=k[\boldsymbol{u}]/I$ be a presentation as in 1.3, with variables $\boldsymbol{u}=u_1,\dots,u_e$. Since $K(\boldsymbol{u};N)$ is a minimal free resolution of k over $k[\boldsymbol{u}]$, we have $\operatorname{Tor}^{k[\boldsymbol{u}]}_i(N,k)_{i+n}\cong \mathrm{H}_i(\boldsymbol{u};N)_{i+n}=0$ for all $i\geq 0$ and all $n>s$, and $\operatorname{Tor}^{k[\boldsymbol{u}]}_e(N,k)_{e+s}\cong \mathrm{H}_e(\boldsymbol{u},N)_{e+s}=\{y\in N_s\,|\,\boldsymbol{u}y=0\}=N_s\neq 0$.

We extend next the notion of regularity to local rings. The notation for a local ring is $(R,\mathfrak{m},k)$, where $\mathfrak{m}$ is the maximal ideal and k is the residue field.

1.6. For a local ring $(R,\mathfrak{m},k)$ and an R-module M we denote $\operatorname{gr}(M)$ the associated graded module $\bigoplus_{n=0}^{\infty}\mathfrak{m}^nM/\mathfrak{m}^{n+1}M$ with respect to the $\mathfrak{m}$-adic filtration. Thus, $\operatorname{gr}(R)$ is a graded ring and $\operatorname{gr}(M)$ is a graded $\operatorname{gr}(R)$-module. Note that $\operatorname{gr}(R)$ is a polynomial ring if and only if R is regular.

We define the *regularity* of M over R by the formula

$$\operatorname{reg}_R(M)=\operatorname{reg}_{\operatorname{gr}(R)}\big(\operatorname{gr}(M)\big)\,.$$

and the *polynomial regularity* of M by the formula

$$\operatorname{pol\,reg}(M)=\operatorname{pol\,reg}\big(\operatorname{gr}(M)\big)\,.$$

1.7. As usual, $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion of R. We say that $Q/\mathfrak{a}$ is a *Cohen presentation* of R if $\widehat{R}\cong Q/\mathfrak{a}$, where $(Q,\mathfrak{n},k)$ is a regular local ring and $\mathfrak{a}$ an ideal of Q. If $\mathfrak{a}$ is contained in $\mathfrak{n}^2$, then we say that the presentation is *minimal*. Cohen's Structure Theorem guarantees that such presentations always exist.

The ring $\widehat{R}$ is R-flat, with maximal ideal $\mathfrak{m}\widehat{R}$, hence there are equalities $\operatorname{gr}(\widehat{R})=\operatorname{gr}(R)$ and $\operatorname{gr}(\widehat{M})=\operatorname{gr}(M)$. Thus, the regularity of M over R, as well as the polynomial regularity of M, do not change under $\mathfrak{m}$-adic completion.

Let $\widehat{R}=Q/\mathfrak{a}$ be a Cohen presentation. Since $\operatorname{gr}(R)$ is then a homomorphic image of $\operatorname{gr}(Q)$, which is a polynomial ring, we have:

$$\operatorname{pol\,reg}(M)=\operatorname{pol\,reg}(\operatorname{gr}(M))=\operatorname{reg}_{\operatorname{gr}(Q)}(\operatorname{gr}(M))=\operatorname{reg}_Q(M)\,.$$

1.8. **Example.** The *Loewy length* of an R-module M, denoted $\ell\ell_R(M)$, is the smallest positive integer n for which $\mathfrak{m}^nM=0$. Recall by 1.5 that if $\operatorname{gr}(M)$ has finite length, then $\operatorname{pol\,reg}\big(\operatorname{gr}(M)\big)$ is equal to the largest integer n such that $\operatorname{gr}(M)_n\neq 0$; this is also the largest integer n such that $\mathfrak{m}^nM\neq 0$. For an Artinian R-module M we have thus $\operatorname{pol\,reg}(M)=\ell\ell_R(M)-1$.

1.9. **Example.** Assume that R is a hypersurface, that is, $\widehat{R}$ has a Cohen presentation $\widehat{R}=Q/(\mathfrak{a})$, with $\mathfrak{a}=(a)$ for some $\mathfrak{a}\in\mathfrak{n}$. In this case, $\operatorname{pol\,reg}(R)=\operatorname{mult}(R)-1$. Indeed, it is well-known that $\operatorname{mult}(R)$ is the smallest integer s such that $a\in\mathfrak{n}^s$. By 1.7 the invariant $\operatorname{pol\,reg}(R)$ is equal to the Castelnuovo-Mumford regularity of

$\mathrm{gr}(R)$ over the polynomial ring $\mathrm{gr}(Q)$. If a^* denotes the initial form of a in $\mathrm{gr}(R)$, then $\mathrm{gr}(R) = \mathrm{gr}(Q)/(a^*)$, hence

$$0 \longrightarrow \mathrm{gr}(Q)(-s) \xrightarrow{a^*} \mathrm{gr}(Q) \longrightarrow 0$$

is a minimal graded free resolution of $\mathrm{gr}(R)$ over $\mathrm{gr}(Q)$, so $\operatorname{pol}\operatorname{reg}(R) = s - 1$.

2. Linear resolutions

Let $(R, \mathfrak{m}, k)$ be a local ring and set $G = \mathrm{gr}(R)$. An R-module M is said to have a *linear resolution* if $\operatorname{reg}_R(M) = 0$. We describe next constructions that will be used to characterize such modules.

2.1. Let X be a complex of R-modules filtered by subcomplexes

$$X = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^p \supseteq \cdots .$$

The associated graded complex $(\mathrm{gr}_F(X), d)$ has

$$\big(\mathrm{gr}_F(X)\big)_{i,p} = F^p_i / F^{p+1}_i$$

and the differential d is induced by ∂.

If the complex X is minimal, that is $\partial(X) \subseteq \mathfrak{m}X$, then it has a filtration defined by $F^p_i = \mathfrak{m}^{p-i} X_i$ for each i, that is:

$$F^p = \quad \cdots \to X_{p+1} \to X_p \to \mathfrak{m}X_{p-1} \to \cdots \to \mathfrak{m}^{p-1}X_1 \to \mathfrak{m}^p X_0 \to 0\,.$$

In this case we set $\mathrm{E}(X) = \mathrm{gr}_F(X)$ and note that $\mathrm{E}(X)$ is a complex of graded G-modules, with $\mathrm{E}(X)_i = \mathrm{gr}(X_i)(-i)$ for each i, and the differential d is homogeneous of degree 0.

2.2. For an R-module M we choose a minimal free resolution X of M over R. It induces a map $\mathrm{E}(X) \to \mathrm{gr}(M)$. Choose a graded free resolution U of $\mathrm{gr}(M)$ over G and choose a lifting of the identity map on $\mathrm{gr}(M)$ to a morphism of complexes of graded G-modules $\iota^M \colon \mathrm{E}(X) \to U$; this lifting is unique up to homotopy. For each $z \in X_i$ we denote $\overline{z}$ the class of z in $\mathrm{E}(X)_{i,i} = X_i/\mathfrak{m}X_i$. Calculating $\operatorname{Tor}^R_i(M, k)$ from the resolution X and $\operatorname{Tor}^G_i\big(\mathrm{gr}(M), k\big)$ from the resolution U, we define homomorphisms of R-modules λ^M_i as follows:

$$\lambda^M_i \colon \operatorname{Tor}^R_i(M, k) = X_i \otimes_R k \longrightarrow \mathrm{H}_i(U \otimes_G k)_i = \operatorname{Tor}^G_i\big(\mathrm{gr}(M), k\big)_i$$
$$\lambda^M_i(z \otimes_R 1) = \operatorname{cls}\big(\iota^M(\overline{z}) \otimes_G 1\big)\,.$$

We then define $\tau^M_i \colon \operatorname{Tor}^R_i(M, k) \to \operatorname{Tor}^G_i\big(\mathrm{gr}(M), k\big)$ to be the composition of λ^M_i with the inclusion $\operatorname{Tor}^G_i\big(\mathrm{gr}(M), k\big)_i \hookrightarrow \operatorname{Tor}^G_i\big(\mathrm{gr}(M), k\big)$.

2.3. **Proposition.** *Let $(R, \mathfrak{m}, k)$ be a local ring, M a finite R-module, and X a minimal free resolution of M over R. The following properties are equivalent:*

1. *The module M has a linear resolution.*
2. $\mathrm{E}(X)$ *is a minimal free resolution of* $\mathrm{gr}(M)$ *over G.*
3. $\mathrm{E}(X)$ *is a free resolution of* $\mathrm{gr}(M)$ *over G.*
4. *For each i the maps τ^M_i and λ^M_i are bijective.*
5. *For each i the map τ^M_i is surjective.*

The content of the proposition is more or less known; see also [16]. Due to lack of a proper reference, we provide a proof, based on the next lemma.

For each integer j we denote $R\{j\}$ the R-module R, filtered by $F^p = \mathfrak{m}^{p-j}$.

2.4. **Lemma.** *Let $(R,\mathfrak{m},k)$ be a local ring and M a finite R-module. There exists a filtered resolution X' of M with $X'_i=\bigoplus_{j=1}^{s_i} R\{a_{ij}\}$ for each $i\geq 0$, such that the associated graded complex is a minimal graded free resolution of $\operatorname{gr}(M)$ over G.*

Proof. For an element $m\in M$ we denote m^* its initial form in $\operatorname{gr}_{\mathfrak{m}}(M)$ and $\operatorname{ord}(m)$ the order of m^*. Let $m_1^*,\dots,m_s^*$ be a minimal system of homogeneous generators of $\operatorname{gr}_{\mathfrak{m}}(M)$ over G and set $a_{0j}=\operatorname{ord}(m_j)$. By [34, II-6, Cor. 2], the elements $m_1,\dots,m_s$ generate M. Set $X'_0=\oplus_{j=1}^s R\{a_{0j}\}$ and let $\partial_0\colon X'_0\to M$ be the surjection that sends to m_j the basis element corresponding to $R\{a_{0j}\}$. The associated graded module of X'_0 is $U_0=\oplus_{j=1}^s G\big(-a_{0j}\big)$, and the map $d_0\colon U_0\to\operatorname{gr}(M)$ induced by ∂_0 is homogeneous and surjective. We set $M_0=\operatorname{Ker}(\partial_0)$ and consider the filtration F_0 induced by X'_0. Note that $\operatorname{gr}_{F_0}(X'_0)=\operatorname{Ker}(d_0)$. The resolution X' is obtained by iterating this procedure. □

Proof of Proposition 2.3. The implications (2) ⇒ (3) and (4) ⇒ (5) are clear.

(1) ⇒ (2) Let X' be the filtered resolution of M from Lemma 2.4. The associated graded complex has $U_i=\bigoplus_{j=1}^{s_i} G(-a_{ij})$. Since $\operatorname{reg}_R(M)=0$, we get $a_{ij}=i$ for all i and j, hence the filtration of X' is given by $F_i^p=\mathfrak{m}^{p-i}X'_i$. The compatibility of this filtration with the differential implies that X' is a minimal free resolution. Any two minimal free resolutions of M over R are isomorphic, and such an isomorphism preserves the filtration described above, hence $U=\mathrm{E}(X')\cong\mathrm{E}(X)$.

(3) ⇒ (4) Both $\mathrm{E}(X)$ and U are graded free resolutions of $\operatorname{gr}(M)$ over G, so $\mathrm{H}_*(\iota^M\otimes_G k)$ is bijective. Since the complex $\mathrm{E}(X)$ is minimal, we have $\mathrm{H}_*\big(\mathrm{E}(X)\otimes_G k\big)=\mathrm{E}(X)\otimes_G k$. The map $\nu_i\colon X_i\otimes_R k\to\big(\mathrm{E}(X)\otimes_G k\big)_{i,i}$ defined by $\nu_i(z\otimes_R 1)=\overline{z}\otimes_G 1$ is clearly bijective. Since $\lambda_i^M=\mathrm{H}_i(\iota^M\otimes_G k)_i\circ\nu_i$ for each i, the map λ_i^M is bijective, and so is τ_i^M, because $\operatorname{Tor}_i^G(\operatorname{gr}(M),k)_p\cong\big(\mathrm{E}(X)\otimes_G k\big)_{i,p}=0$ for $i\neq p$.

(5) ⇒ (1). The image of the map τ_i^M is in $\operatorname{Tor}_i^G(\operatorname{gr}(M),k)_i$. If τ_i^M is surjective for each i, then $\operatorname{Tor}_i^G(\operatorname{gr}(M),k)_p=0$ for $p\neq i$, hence $\operatorname{reg}_R(M)=0$. □

We defined the homomorphisms $\lambda_i^M\colon\operatorname{Tor}_i^R(M,k)\to\operatorname{Tor}_i^G\big(\operatorname{gr}(M),k\big)_i$ in terms of free resolutions of M and respectively $\operatorname{gr}(M)$. We show next that one can also compute these maps using free resolutions of k. To do so, we take a more general point of view.

2.5. Let X be a minimal complex of free R-modules. Since $\mathrm{E}(X)_{i,p}=0$ for $p<i$ and $\mathrm{E}_{i,i}=X_i/\mathfrak{m}X_i$, we have

$$\mathrm{H}_i\big(\mathrm{E}(X)\big)_i=\mathrm{Z}_i\big(\mathrm{E}(X)\big)_i\subseteq X_i/\mathfrak{m}X_i\,.$$

Recall that for each $z\in X_i$ we denote $\overline{z}$ its class in $\mathrm{E}(X)_{i,i}$. We then define canonical homomorphisms

$$\nu_i^X\colon\ \mathrm{H}_i(X)\to\mathrm{H}_i\big(\mathrm{E}(X)\big)_i\,,\quad\text{by setting}\quad \nu_i^X\big(\operatorname{cls}(z)\big)=\overline{z}\,.$$

If X and Y are two minimal complexes of R-modules, then an easy calculation shows that $\mathrm{E}(X\otimes_R Y)=\mathrm{E}(X)\otimes_G\mathrm{E}(Y)$ as complexes of graded G-modules, hence for each i we have a map

$$\nu_i^{X\otimes_R Y}\colon\ \mathrm{H}_i(X\otimes_R Y)\to\mathrm{H}_i\big(\mathrm{E}(X)\otimes_G\mathrm{E}(Y)\big)_i\,.$$

Let M and N be two R-modules. We consider them as complexes concentrated in degree 0 and note that $\mathrm{E}(M)=\operatorname{gr}(M)$ and $\mathrm{E}(N)=\operatorname{gr}(N)$. Let $\kappa^X: X\to M$ be a minimal free resolution of M over R and $\kappa^Y: Y\to N$ a minimal free resolution of N over R. They induce morphisms $\mathrm{E}(\kappa^X)\colon\ \mathrm{E}(X)\to\operatorname{gr}(M)$ and $\mathrm{E}(\kappa^Y)\colon\ \mathrm{E}(Y)\to$

$\mathrm{gr}(N)$. For each graded free resolution $\kappa^U : U \to \mathrm{gr}(M)$ of $\mathrm{gr}(M)$ over G and each graded free resolution $\kappa^V : V \to \mathrm{gr}(N)$ of $\mathrm{gr}(N)$ over G we consider morphisms of graded complexes $\iota^M\colon\ \mathrm{E}(X) \to U$ and $\iota^N\colon\ \mathrm{E}(Y) \to V$ as in 2.2; these morphisms are unique up to homotopy. For each integer i we set

$$\alpha_{i,i} = \mathrm{H}_i\left(\mathrm{E}(X) \otimes_G \mathrm{E}(\kappa^Y)\right)_i \quad \text{and} \quad \beta_{i,i} = \mathrm{H}_i\left(\mathrm{E}(\kappa^X) \otimes_G \mathrm{E}(Y)\right)_i .$$

The commutative diagram below defines maps $\lambda_i(M,N)$ as indicated:

$$\begin{array}{ccccc}
\mathrm{Tor}_i^R(M,N) & & \xrightarrow{\qquad\qquad\lambda_i(M,N)\qquad\qquad} & & \mathrm{Tor}_i^G\left(\mathrm{gr}(M),\mathrm{gr}(N)\right)_i \\
\Vert & & & & \Vert \\
\mathrm{H}_i(X \otimes_R N) & \xrightarrow{\nu_i^{X\otimes_R N}} & \mathrm{H}_i(\mathrm{E}(X) \otimes_G \mathrm{gr}(N))_i & \xrightarrow{\mathrm{H}_i(\iota^M \otimes_G \mathrm{gr}(N))_i} & \mathrm{H}_i(U \otimes_G \mathrm{gr}(N))_i \\
{\scriptstyle \mathrm{H}_i(X\otimes_G\kappa^Y)}\big\uparrow\cong & & \big\uparrow{\scriptstyle \alpha_{i,i}} & & {\scriptstyle \mathrm{H}_i(U\otimes_G\kappa^V)_i}\big\uparrow\cong \\
\mathrm{H}_i(X \otimes_R Y) & \xrightarrow{\nu_i^{X\otimes_R Y}} & \mathrm{H}_i(\mathrm{E}(X) \otimes_G \mathrm{E}(Y))_i & \xrightarrow{\mathrm{H}_i(\iota^M \otimes_R \iota^N)_i} & \mathrm{H}_i(U \otimes_G V)_i \\
{\scriptstyle \mathrm{H}_i(\kappa^X\otimes_R Y)}\big\downarrow\cong & & \big\downarrow{\scriptstyle \beta_{i,i}} & & {\scriptstyle \mathrm{H}_i(\kappa^U\otimes_G V)_i}\big\downarrow\cong \\
\mathrm{H}_i(M \otimes_R Y) & \xrightarrow{\nu_i^{M\otimes_R Y}} & \mathrm{H}_i(\mathrm{gr}(M) \otimes_G \mathrm{E}(Y))_i & \xrightarrow{\mathrm{H}_i(\mathrm{gr}(M) \otimes_G \iota^N)_i} & \mathrm{H}_i(\mathrm{gr}(M) \otimes_G V)_i
\end{array}$$

We note that $\lambda_i(M,k)$ coincides with the map λ_i^M defined at the beginning of the section. Thus, these maps can be calculated from any horizontal line of the diagram above, with $N = k$.

3. Indices of modules

Each homomorphism of R-modules $\psi\colon\ M \to N$ induces homomorphisms of graded vector spaces

$$\begin{aligned}
\mathrm{Tor}_*^R(\psi,k)\colon\ \mathrm{Tor}_*^R(M,k) &\to \mathrm{Tor}_*^R(N,k)\,,\\
\mathrm{Ext}_R^*(\psi,k)\colon\ \mathrm{Ext}_R^*(N,k) &\to \mathrm{Ext}_R^*(M,k)\,,\\
\mathrm{Ext}_R^*(k,\psi)\colon\ \mathrm{Ext}_R^*(k,M) &\to \mathrm{Ext}_R^*(k,N)\,.
\end{aligned}$$

Let n be an integer. For each R-module M consider the canonical inclusion

$$\mu_M^{(n)} \colon \mathfrak{m}^n M \to \mathfrak{m}^{n-1} M\,.$$

If $\varphi\colon\ (Q,\mathfrak{n}) \to (R,\mathfrak{m})$ is a surjective homomorphism of local rings, then $\mathfrak{m}^n M = \mathfrak{n}^n M$, so the notation $\mu_M^{(n)}$ will be used without reference to a specific ring.

3.1. We define the *Levin index* of M over R by the formula:

$$\boldsymbol{L}_R(M) = \inf\{s \geq 1 \mid \mathrm{Tor}_*^R(\mu_M^{(n)},k) = 0 \ \text{ for all } n \geq s\}\,.$$

The isomorphism of functors $\mathrm{Ext}_R^*(-,k) \cong \mathrm{Hom}_k\left(\mathrm{Tor}_*^R(-,k),k\right)$ shows that

$$\boldsymbol{L}_R(M) = \inf\{s \geq 1 \mid \mathrm{Ext}_R^*(\mu_M^{(n)},k) = 0 \ \text{ for all } n \geq s\}\,.$$

Results of Levin [21] show that $\boldsymbol{L}_R(M) < \infty$, cf. 3.6 for more details.

We define the *Roos index* of M over R by the formula

$$\boldsymbol{R}_R(M) = \inf\{s \geq 1 \mid \mathrm{Ext}_R^*(k,\mu_M^{(n)}) = 0 \ \text{ for all } n \geq s\}\,.$$

Roos [27] noted that Levin's arguments can be adapted to show that $\boldsymbol{R}_R(M) < \infty$, cf. 3.7 for details.

Following [16], we say that a ring R is *Koszul* if its residue field has a linear resolution, that is, $\operatorname{reg}_R(k) = 0$; recall that the notions of regularity and polynomial regularity of local rings were discussed in the first section.

3.2. **Theorem.** *If $(R, \mathfrak{m})$ is a Koszul local ring and M a finite R-module, then*

(1) $\operatorname{reg}_R(\mathfrak{m}^i M) = \max\{\operatorname{reg}_R(M) - i, 0\}$ *for all* $i \geq 0$.
(2) $\boldsymbol{L}_R(M) = \operatorname{reg}_R(M) + 1$.

3.3. **Theorem.** *For a finite module M over an arbitrary local ring R there is an inequality*

$$\max\{\boldsymbol{L}_R(M), \boldsymbol{R}_R(M)\} \leq \operatorname{pol}\operatorname{reg}(M) + 1\,.$$

We postpone the proofs for the moment, in order to give an application of Theorem 3.3.

The *Poincaré series* of a finite R-module M is the formal power series

$$\mathrm{P}^R_M(t) = \sum_{i=0}^{\infty} \operatorname{rank}_k \operatorname{Tor}^R_i(M, k) t^i\,.$$

The *Bass series* of M is the formal power series

$$\mathrm{I}^M_R(t) = \sum_{i=0}^{\infty} \operatorname{rank}_k \operatorname{Ext}^i_R(k, M) t^i\,.$$

The *Hilbert series* of M is the formal power series

$$\operatorname{Hilb}^R_M(t) = \sum_{i=0}^{\infty} \operatorname{rank}_k(\mathfrak{m}^i M/\mathfrak{m}^{i+1} M) t^i$$

The following corollary contains effective versions of [6, 4.1.8], [21, Theorem 2] (see also [6, 6.3.6]); we also include versions for Bass series.

3.4. **Corollary.** *Set $p = \operatorname{pol}\operatorname{reg}(M)$. For each submodule M' contained in $\mathfrak{m}^{p+1}M$ and for each integer $n > p$ the following hold:*

$$\mathrm{P}^R_{M/M'}(t) = \mathrm{P}^R_M(t) + t\,\mathrm{P}^R_{M'}(t) \qquad \mathrm{I}^{M/M'}_R(t) = \mathrm{I}^M_R(t) + t\,\mathrm{I}^{M'}_R(t)\,.$$
$$\mathrm{P}^R_{\mathfrak{m}^n M}(t) = \operatorname{Hilb}^R_{\mathfrak{m}^n M}(-t)\,\mathrm{P}^R_k(t) \qquad \mathrm{I}^{\mathfrak{m}^n M}_R(t) = \operatorname{Hilb}^R_{\mathfrak{m}^n M}(-t)\,\mathrm{I}^k_R(t)\,.$$

Proof. Let α be the inclusion $M' \hookrightarrow M$, let β be the inclusion $M' \hookrightarrow \mathfrak{m}^{p+1}M$, and γ the inclusion $\mathfrak{m}^{p+1}M \hookrightarrow M$. We have then $\alpha = \gamma \circ \mu^{(p+1)} \circ \beta$, hence

$$\operatorname{Tor}^R_*(\alpha, k) = \operatorname{Tor}^R_*(\gamma, k) \circ \operatorname{Tor}^R_*(\mu^{(p+1)}, k) \circ \operatorname{Tor}^R_*(\beta, k)\,.$$

Theorem 3.3 gives $\operatorname{Tor}^R_*(\mu^{(p+1)}_M, k) = 0$, hence $\operatorname{Tor}^R_*(\alpha, k) = 0$. Consider the short exact sequence

$$0 \to M' \xrightarrow{\alpha} M \xrightarrow{\pi} M/M' \to 0\,.$$

The long exact sequence obtained by applying $\operatorname{Tor}^R_*(-, k)$ splits into short exact sequences. Computing ranks, we obtain the equality $\mathrm{P}^R_{M/M'}(t) = \mathrm{P}^R_M(t) + t\,\mathrm{P}^R_{M'}(t)$. The corresponding equality for Bass series is obtained similarly.

The expression for $\mathrm{P}^R_{\mathfrak{m}^n M}(t)$ can be deduced using the calculations in [21, Theorem 2] or [6, 6.3.6]. Similar computations apply to Bass series. □

Proof of theorem 3.2. We choose a minimal free resolution Y of k over R. By Proposition 2.3 the complex $\mathrm{E}(Y)$ is a minimal free resolution of k over G. For all integers i and n we have

$$\operatorname{Tor}_i^G(\operatorname{gr}(M),k) \cong \mathrm{H}_i\left(\operatorname{gr}(M)\otimes_G \mathrm{E}(Y)\right) = \mathrm{H}_i\left(\mathrm{E}(M\otimes_R Y)\right),$$
$$\operatorname{Tor}_i^G\left(\operatorname{gr}(\mathfrak{m}^n M),k\right) \cong \mathrm{H}_i\left(\operatorname{gr}(\mathfrak{m}^n M)\otimes_G \mathrm{E}(Y)\right) = \mathrm{H}_i\left(\mathrm{E}(\mathfrak{m}^n M\otimes_R Y)\right).$$

Since $\mathrm{E}(\mathfrak{m}^n M\otimes_R Y)_{i,p} = \mathrm{E}(M\otimes_R Y)_{i,p+n}$ we further have

$$\mathrm{H}_i\left(\mathrm{E}(\mathfrak{m}^n M\otimes_R Y)\right)_p \cong \begin{cases} \mathrm{H}_i\left(\mathrm{E}(M\otimes_R Y)\right)_{p+n} & \text{if} \quad p>i \\ \mathrm{Z}_i\left(\mathrm{E}(M\otimes_R Y)\right)_{p+n} & \text{if} \quad p=i\,. \end{cases}$$

These isomorphisms prove (1). In particular, we have $\operatorname{reg}_R(\mathfrak{m}^n M)=0$ for $n\geq \operatorname{reg}_R(M)$, hence the maps $\lambda_i^{\mathfrak{m}^n M}$ are bijective by Proposition 2.3(4). As shown in 2.5, we have $\lambda_i^{\mathfrak{m}^n M} = \lambda_i(\mathfrak{m}^n M,k)$, and this map can be computed from the last row of the diagram considered there, as the composition $\mathrm{H}_i\left(\operatorname{gr}(\mathfrak{m}^n M)\otimes_G \iota^k\right)_i \circ \nu_i^{\mathfrak{m}^n M\otimes_R Y}$. Since both V and $\mathrm{E}(Y)$ are graded free resolutions of k over G, the first map is an isomorphism. It follows that $\nu_i^{\mathfrak{m}^n M\otimes_R Y}$ is bijective for all i and all $n\geq \operatorname{reg}_R(M)$. To simplify notation, we denote these maps $\nu_i^{(n)}$ and we recall their definition:

$$\nu_i^{(n)}\colon\ \mathrm{H}_i(\mathfrak{m}^n M\otimes_R Y)\to \mathrm{H}_i\big(\mathrm{E}(\mathfrak{m}^n M\otimes_R Y)\big)_i \subseteq \frac{\mathfrak{m}^n M\otimes_R Y_i}{\mathfrak{m}^{n+1}M\otimes_R Y_i}\,,$$
$$\nu_i^{(n)}(\operatorname{cls}(z)) = z + (\mathfrak{m}^{n+1}M\otimes_R Y_i)\,.$$

We set $l=\boldsymbol{L}_R(M)$ and $r=\operatorname{reg}_R(M)$. To prove the inequality $r+1\geq l$, we show that the map

$$\mathrm{H}(\mu_M^{(n)}\otimes_R Y)\colon\ \mathrm{H}(\mathfrak{m}^n M\otimes_R Y)\to \mathrm{H}(\mathfrak{m}^{n-1}M\otimes_R Y)$$

is zero for all $n\geq r+1$. Let $z\in \mathfrak{m}^n M\otimes_R Y$ be a cycle of degree i, with $n\geq r+1$. We regard z as a cycle of the subcomplex $\mathfrak{m}^{n-1}M\otimes_R Y$. Since $\nu_i^{(n-1)}$ is bijective and $\nu_i^{(n-1)}(\operatorname{cls}(z))=0$, it follows that z is a boundary in $\mathfrak{m}^{n-1}M\otimes_R Y$.

We assume next that $r+1>l$. We want to prove that $\operatorname{Tor}_i^G\left(\operatorname{gr}(M),k\right)_{i+r}=0$ for all i, which contradicts $r=\operatorname{reg}_R(M)$. By the above we have

$$\operatorname{Tor}_i^G\left(\operatorname{gr}(M),k\right)_{i+r} \cong \mathrm{H}_i\left(\mathrm{E}(M\otimes_R Y)\right)_{i+r}.$$

Let z be an element of $\mathfrak{m}^r M\otimes_R Y_i$ whose image $\overline{z}$ in $\mathrm{E}(M\otimes_R Y)_{i,i+r}$ is a cycle. We need to show that $\overline{z}$ is a boundary of $\mathrm{E}(M\otimes_R Y)$. We regard $\overline{z}$ as a cycle of $\mathrm{E}(\mathfrak{m}^r M\otimes_R Y)$. Since the map $\nu_i^{(r)}$ is bijective, there exists a cycle $z'\in \mathfrak{m}^r M\otimes_R Y_i$ such that $\overline{z}=\overline{z}'$ in $\mathrm{E}(\mathfrak{m}^r M\otimes_R Y)_{i,i} = \mathrm{E}(M\otimes_R Y)_{i,i+r}$. The assumption that $r\geq l$ and the definition of l imply that the map $\mathrm{H}(\mu_M^{(r)}\otimes_R Y)\colon\ \mathrm{H}(\mathfrak{m}^r M\otimes_R Y)\to \mathrm{H}(\mathfrak{m}^{r-1}M\otimes_R Y)$ is zero, hence z' is a boundary of the complex $\mathfrak{m}^{r-1}M\otimes_R Y$. It follows that $\overline{z}'$, and hence $\overline{z}$, is a boundary of $\mathrm{E}(M\otimes_R Y)$. □

3.5. For an arbitrary local ring $(R,\mathfrak{m})$ we choose a minimal system of generators $\boldsymbol{g}$ of $\mathfrak{m}$ and set $K^R=K(\boldsymbol{g};R)$. If M is an R-module, then we set $K(M)=K(\boldsymbol{g};M)$ and $\mathrm{H}_*(M)=\mathrm{H}_*\left(K(M)\right)$. For an R-homomorphism $\psi\colon M\to M'$ we denote $\mathrm{H}_*(\psi)$ the induced map $\mathrm{H}_*(M)\to \mathrm{H}_*(M')$. If $\varphi\colon R\to S$ is a surjective homomorphism of local rings with $\operatorname{Ker}(\varphi)\subseteq \mathfrak{m}^2$, then $\varphi(\boldsymbol{g})$ is a minimal system of generators of the maximal ideal of S. Thus, we can identify K^S and $K^R\otimes_R S$. If the R-module structure of M is induced through the homomorphism φ, then, as in 1.1, we identify $K(\varphi(\boldsymbol{g});M)$ and $K(\boldsymbol{g},M)$ and let the notation $K(M)$ refer to either complex.

The complex $\operatorname{Hom}_R(K^R,M)$ is known to be isomorphic to $K(M)$, cf. [10, 1.6.10]. We denote $\mathrm{H}^*(M)$ its homology. Also, we denote $\mathrm{H}^*(\varphi)\colon \mathrm{H}^*(M')\to\mathrm{H}^*(M)$ the map induced by the homomorphism $\varphi\colon M\to M'$, which is the k-dual of $\mathrm{H}_*(\varphi)$, and we make identifications similar to those above.

3.6. Levin [21, Lemma 1] proves that $\mathrm{H}_*\left(\mu_M^{(n)}\right)$ for all large n. Also, [21, Lemma 2] shows that $\mathrm{H}_*\left(\mu_M^{(n)}\right)=0$ implies $\operatorname{Tor}_*^R\left(\mu_M^{(n)},k\right)=0$. In particular, $\boldsymbol{L}_R(M)<\infty$.

3.7. Since the map $\mathrm{H}^*\left(\mu_M^{(n)}\right)$ is dual to $\mathrm{H}_*\left(\mu_M^{(n)}\right)$, it is also zero for all large n. As noted by Roos [27], techniques similar to those of Levin show that $\mathrm{H}^*\left(\mu_M^{(n)}\right)=0$ implies $\operatorname{Ext}_R^*\left(k,\mu_M^{(n)}\right)=0$. In particular, $\boldsymbol{R}_R(M)<\infty$.

Proof of Theorem 3.3. Since $\widehat{R}$ is R-flat, with maximal ideal $\mathfrak{m}\widehat{R}$, there are equalities $\boldsymbol{L}_R(M)=\boldsymbol{L}_{\widehat{R}}(\widehat{M})$ and $\boldsymbol{R}_R(M)=\boldsymbol{R}_{\widehat{R}}(\widehat{M})$. Thus, we can assume that R has a minimal Cohen presentation $R=Q/\mathfrak{a}$. Since Q is regular, the Koszul complex K^Q is a minimal free resolution of k over Q, so $\operatorname{Tor}_*^Q(-,k)=\mathrm{H}_*(-)$ and $\operatorname{Ext}_Q^*(k,-)=\mathrm{H}^*(-)$, in the notation of 3.5. For any Q-module there are isomorphisms $\mathrm{H}_i(N)\cong\mathrm{H}^{d-i}(N)$, where $d=\dim(Q)$, cf. [10, 1.6.10]. We have thus:

$$\begin{aligned}\boldsymbol{L}_Q(M)&=\inf\{s\geq 1\mid \mathrm{H}_*\left(\mu_M^{(n)}\right)=0 \text{ for all } n\geq s\}\\&=\inf\{s\geq 1\mid \mathrm{H}^*\left(\mu_M^{(n)}\right)=0 \text{ for all } n\geq s\}=\boldsymbol{R}_Q(M)\,.\end{aligned}$$

By 3.5 the maps $\mathrm{H}_*\left(\mu_M^{(n)}\right)$ and $\mathrm{H}^*\left(\mu_M^{(n)}\right)$ do not depend on whether M is viewed as a module over R or over Q. In our notation, 3.6 and 3.7 translate as follows:

$$\max\{\boldsymbol{L}_R(M),\boldsymbol{R}_R(M)\}\leq\boldsymbol{L}_Q(M)=\boldsymbol{R}_Q(M)<\infty\,.$$

Note that Q is a Koszul local ring, since $\operatorname{gr}(Q)$ is a polynomial ring over k, hence a minimal graded free resolution of k over this ring is given by the Koszul complex on the variables. By 3.2 we have then $\boldsymbol{L}_Q(M)=\operatorname{reg}_Q(M)+1$. To finish the proof, recall from 1.7 that $\operatorname{reg}_Q(M)=\operatorname{pol}\operatorname{reg}(M)$. □

4. Higher delta invariants

For a finite module M over a Gorenstein local complete ring $(R,\mathfrak{m})$ Auslander defined the *delta invariant* $\delta_R(M)$ to be the smallest integer n such that there exists an epimorphism $X\oplus R^n\to M$, where X is a maximal Cohen Macaulay module with no free summand. For an integer $i\geq 0$ he defined an i-th *higher delta invariant* $\delta_R^i(M)$ by the formula $\delta_R^i(M)=\delta_R(\Omega_R^i(M))$, where $\Omega_R^i(M)$ denotes the i-th syzygy module in a minimal free resolution of M over R, cf. [2, §5].

If R is not regular, then Auslander proved $\delta_R^i(k)=0$ for all $i>0$, cf. [2, 5.7]. Yoshino [39] studied the vanishing of the numbers $\delta_R^i(R/\mathfrak{m}^n)$ for positive integers i and n. He conjectured that if R is not regular, then they all vanish. One of his main results [39, (2.1)] shows that there exists an integer s such that $\delta_R^i(R/\mathfrak{m}^n)=0$ for all $n\geq s$ and all $i>0$, or, equivalently, $\delta_R^i(\mathfrak{m}^n)=0$ for all $n\geq s$ and all $i\geq 0$.

Buchweitz has noted, cf. [26, 2.3], that $\delta_R^i(M)$ equals the rank of the kernel of

$$\varepsilon_R^i(M,k)\colon \operatorname{Ext}_R^i(M,k)\to\check{\operatorname{Ext}}{}_R^i(M,k)\,,$$

where $\varepsilon^*_R(-,-)$ is the natural transformation from absolute cohomology to Tate cohomology. The latter cohomology theory is defined for all modules over a Gorenstein ring, but not in general. Over an arbitrary ring, Vogel has defined a cohomology theory, $\check{\mathrm{Ext}}^*_R(-,-)$, which comes with a natural transformation $\varepsilon^*_R(-,-)$ as above, and coincides with Tate cohomology when R is Gorenstein. Vogel's theory is described in [11], cf. also [24].

Martsinkovsky [24] sets $\xi^i_R(M) = \operatorname{rank}_k \operatorname{Ker}\big(\varepsilon^i_R(M,k)\big)$ and proves:

4.1. ([25, Theorem 6]). If R is not regular, then $\xi^i_R(k) = 0$ for all i.

We use the cohomological interpretation of delta invariants to extend Yoshino's result to arbitrary local rings, to generalize it to submodules $\mathfrak{m}^n M$ of any finite R-module, and to obtain bounds for the vanishing of $\delta^i_R(\mathfrak{m}^n M)$.

4.2. **Theorem.** *If M is a finite module over a non-regular local ring $(R,\mathfrak{m},k)$, then the homomorphism $\varepsilon^i_R(\mathfrak{m}^n M,k)$ is injective for all $i \geq 0$ and all $n \geq \boldsymbol{L}_R(M)$. In particular, if R is Gorenstein, then $\delta^i_R(\mathfrak{m}^n M) = 0$ for all $n > \operatorname{pol\,reg}(M)$.*

Proof. For each integer n we set $M_n = \mathfrak{m}^{n-1}M/\mathfrak{m}^n M$ and form the exact sequence

$$0 \to \mathfrak{m}^n M \xrightarrow{\mu^{(n)}_M} \mathfrak{m}^{n-1}M \to M_n \to 0\,.$$

It induces long exact sequences in cohomology, both for Ext and $\check{\mathrm{Ext}}$. The naturality of $\varepsilon^*_R(-,k)$ implies that for each i there is a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Ext}^i_R(\mathfrak{m}^{n-1}M,k) & \xrightarrow{\mathrm{Ext}^i_R(\mu^{(n)}_M,k)} & \mathrm{Ext}^i_R(\mathfrak{m}^n M,k) & \xrightarrow{\eth^i} & \mathrm{Ext}^{i+1}_R(M_n,k) \\
\Big\downarrow{\scriptstyle \varepsilon^i_R(\mathfrak{m}^{n-1}M,k)} & & \Big\downarrow{\scriptstyle \varepsilon^i_R(\mathfrak{m}^n M,k)} & & \Big\downarrow{\scriptstyle \varepsilon^{i+1}_R(M_n,k)} \\
\check{\mathrm{Ext}}^i_R(\mathfrak{m}^{n-1}M,k) & \xrightarrow{\check{\mathrm{Ext}}^i_R(\mu^{(n)}_M,k)} & \check{\mathrm{Ext}}^i_R(\mathfrak{m}^n M,k) & \xrightarrow{\check{\eth}^i} & \check{\mathrm{Ext}}^{i+1}_R(M_n,k)
\end{array}$$

where $\eth^i$ and $\check{\eth}^i$ denote connecting homomorphisms. If $n \geq \boldsymbol{L}_R(M)$, then we have $\mathrm{Ext}^i_R(\mu^{(n)}_M,k) = 0$ by the definition in 3.1, hence $\eth^i$ is injective. Since $\mathfrak{m}M_n = 0$, the map $\varepsilon^{i+1}_R(M_n,k)$ can be identified with the natural map $\varepsilon^{i+1}_R(k,k) \otimes_R \operatorname{Hom}_k(M_n,k)$. The map $\varepsilon^{i+1}_R(k,k)$ is injective by 4.1, hence so is $\varepsilon^{i+1}_R(M_n,k)$. The commutativity of the right-hand square implies that $\varepsilon^i_R(\mathfrak{m}^n M,k)$ is injective, hence $\delta^i_R(\mathfrak{m}^n M) = 0$. The last statement of the theorem follows by 3.3. □

5. Small homomorphisms

Let $(R,\mathfrak{m},k)$ be a local ring and let $\varphi\colon R \to S$ be a surjective homomorphism of rings. Due to the functoriality of Tor and Ext in the ring variable there are homomorphisms of graded vector spaces

$$\begin{aligned}
\mathrm{Tor}^{\varphi}_*(k,k)\colon\ & \mathrm{Tor}^R_*(k,k) \to \mathrm{Tor}^S_*(k,k) \\
\mathrm{Ext}^*_{\varphi}(k,k)\colon\ & \mathrm{Ext}^*_S(k,k) \to \mathrm{Ext}^*_R(k,k)\,.
\end{aligned}$$

Recall that $\mathrm{Ext}^*_{\varphi}(k,k)$ is a homomorphism of k-graded algebras, where multiplication on the Ext's is given by the Yoneda products.

Following Avramov [3], we say that a surjective homomorphism $\varphi\colon R \to S$ is *small* if $\mathrm{Tor}^{\varphi}_*(k,k)$ is injective, or, equivalently, if the algebra homomorphism $\mathrm{Ext}^*_{\varphi}(k,k)$ is surjective.

For each integer n we consider the canonical homomorphism

$$\rho^{(n)}\colon R \to R/\mathfrak{m}^n\,.$$

By [3, 4.1] the homomomorphism $\rho^{(n)}$ is small for all large n.

5.1. We define the *Avramov index* by the formula

$$\boldsymbol{A}(R) = \inf\{s \geq 0 \mid \rho^{(s+1)} \text{ is small}\}\,.$$

We note that if $\rho^{(s)}$ is small for some integer s, then $\rho^{(n)}$ is small for all $n \geq s$. Indeed, if $\upsilon\colon R/\mathfrak{m}^n \to R/\mathfrak{m}^s$ is the induced map, then the functoriality of Tor gives

$$\operatorname{Tor}_*^{\rho^{(s)}}(k,k) = \operatorname{Tor}_*^{\upsilon}(k,k) \circ \operatorname{Tor}_*^{\rho^{(n)}}(k,k)\,.$$

Thus, the definition of $\boldsymbol{A}(R)$ can be reformulated in terms similar to those of the other indices:

$$\boldsymbol{A}(R) = \inf\{s \geq 0 \mid \rho^{(n)} \text{ is small for all } n > s\}\,.$$

5.2. By [3, 3.9] a homomorphism φ is small if and only if the induced homomorphism $\widehat{\varphi}\colon \widehat{R} \to \widehat{S}$ is small. Thus, $\boldsymbol{A}(R) = \boldsymbol{A}(\widehat{R})$ and we will assume whenever necessary that R is complete, with Cohen presentation $R = Q/\mathfrak{a}$ as in 1.7.

For completeness, we include a proof of the following known result.

5.3. **Proposition.** *For a local ring $(R, \mathfrak{m}, k)$ an inequality $\boldsymbol{A}(R) \leq 1$ holds if and only if $\operatorname{Ext}_R^*(k,k)$ is generated as an algebra by its elements of degree 1. Moreover, $\boldsymbol{A}(R) = 0$ holds if and only if $\mathfrak{m} = 0$.*

Proof. The inequality $\boldsymbol{A}(R) \leq 1$ implies the surjectivity of the algebra homomorphism $\operatorname{Ext}^*_{\rho^{(2)}}(k,k)$. Since $\operatorname{Ext}^*_{R/\mathfrak{m}^2}(k,k)$ is the tensor algebra over k generated by the elements of degree 1, cf. [28, §1, Remark 3], the conclusion follows.

If $R = k$, then it is clear that $\boldsymbol{A}(R) = 0$. Conversely, if $\boldsymbol{A}(R) = 0$, then we have an injection $\operatorname{Tor}_1^{\rho^{(1)}}(k,k)\colon \operatorname{Tor}_1^R(k,k) = \mathfrak{m}/\mathfrak{m}^2 \to \operatorname{Tor}_1^k(k,k) = 0$, hence $\mathfrak{m} = 0$ by Nakayama's Lemma. □

Recall that a ring R is said to be a *complete intersection* if the ideal $\mathfrak{a}$ in some Cohen presentation is generated by a Q-regular sequence. It is known that this notion does not depend on the choice of the presentation, cf. [3, 7.3.3], for example. If the ideal $\mathfrak{a}$ is principal, then R is a *hypersurface*.

5.4. **Proposition.** *Let $(R, \mathfrak{m})$ be a local ring with $\mathfrak{m} \neq 0$ and minimal Cohen presentation $\widehat{R} = Q/\mathfrak{a}$. The following then hold:*

(1) $\boldsymbol{A}(R) \geq \inf\{i \geq 1 \mid \mathfrak{a} \cap \mathfrak{n}^{i+2} \subseteq \mathfrak{n}\mathfrak{a}\}$.
(2) *If R a complete intersection, then equality holds in* (1).
(3) *If R is a hypersurface, then $\boldsymbol{A}(R) = \max\{1, \operatorname{mult}(R) - 1\}$.*
(4) *If $\boldsymbol{x}$ is a regular sequence in R and $\mathfrak{m} \neq (\boldsymbol{x})$, then $\boldsymbol{A}(R) \leq \boldsymbol{A}(R/(\boldsymbol{x}))$.*

The proof is based on results about small homomorphisms from [3].

5.5. A DG *algebra* is a complex (Λ, ∂) with an unitary associative product such that the differential satisfies the Leibnitz rule: $\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)$, where $|a|$ denotes the homological degree of a. In addition, we assume DG algebras to be graded commutative, that is $ab = (-1)^{|a||b|}ba$ for all $a, b \in \Lambda$, and $a^2 = 0$ when $|a|$ is odd. We refer to [6] and [19] for details.

A *system of divided powers* on a DG algebra Λ is an operation that associates to every element $a \in \Lambda$ of even positive degree a sequence of elements $a^{(i)} \in \Lambda$ with

$i = 0, 1, 2, \dots$ satisfying certain axioms, cf. [14, 1.7.1]. A DG Γ-*algebra* is a DG algebra with divided powers which are compatible with the differential, in the sense that $\partial(a^{(i)}) = \partial(a)a^{(i-1)}$ for every $a \in \Lambda$ of positive even degree and every $i \geq 1$.

By Gulliksen [13] and Schoeller [32] there exists a minimal free resolution of k over R which has a structure of DG Γ-algebra; it is obtained by Tate's procedure of adjoining divided powers variables (cf. [38]). We call it a *minimal Tate resolution* of k over R. Note that $\operatorname{Tor}^R_*(k, k)$ inherits a structure of DG Γ-algebra.

Let Λ be a DG Γ-algebra and denote $\Lambda_{>0}$ the ideal of elements of positive degree. The *module of* Γ-*indecomposables* of Λ is the quotient of $\Lambda_{>0}$ by the submodule generated by all elements of the form uv with $u, v \in \Lambda_{>0}$ and $w^{(n)}$ with $w \in \Lambda_{2i}, n \geq 2$. We denote $\pi_*(R)$ the module of Γ-indecomposables of $\operatorname{Tor}^R_*(k, k)$.

The next result is our main tool in the study of small homomorphisms.

5.6. ([3, 3.1]) A surjective homomorphism of local rings $\varphi\colon R \to S$ is small if and only if the induced homomorphism $\pi_*(\varphi)\colon \pi_*(R) \to \pi_*(S)$ is injective.

We proceed to describe the maps $\pi_1(\varphi)$ and $\pi_2(\varphi)$.

5.7. By 5.2 we can consider a minimal Cohen presentation $R = Q/\mathfrak{a}$. We then have $S = Q/\mathfrak{b}$ for an ideal $\mathfrak{b} \supseteq \mathfrak{a}$. Since $\pi_1(R) = \operatorname{Tor}^R_1(k, k)$, the map $\pi_1(\varphi) = \operatorname{Tor}^\varphi_1(k, k)$ is canonically identified with the natural map $\mathfrak{n}/\mathfrak{n}^2 \to \mathfrak{n}/(\mathfrak{b}, \mathfrak{n}^2)$. It is injective if and only if $\mathfrak{b} \subseteq \mathfrak{n}^2$. When this happens, $\pi_1(\varphi)$ is bijective.

Assume that $\mathfrak{b} \subseteq \mathfrak{n}^2$. The proof of [14, Proposition 3.3.4] canonically identifies $\pi_2(R)$ with $\mathfrak{a}/\mathfrak{n}\mathfrak{a}$ and $\pi_2(S)$ with $\mathfrak{b}/\mathfrak{n}\mathfrak{b}$; under these identifications, $\pi_2(\varphi)$ is the natural homomorphism $\mathfrak{a}/\mathfrak{n}\mathfrak{a} \to \mathfrak{b}/\mathfrak{n}\mathfrak{b}$. This map is injective if and only if $\mathfrak{a} \cap \mathfrak{n}\mathfrak{b} \subseteq \mathfrak{n}\mathfrak{a}$, or, equivalently, if a minimal set of generators of $\mathfrak{a}$ can be completed to a minimal set of generators of $\mathfrak{b}$.

The behavior of smallness under factorization of a regular sequence is described by the following result of Tate [38, Theorem 4] and Scheja [33, Satz 1], in the form given by Gulliksen [14, 3.4.1].

5.8. Let $\boldsymbol{x}$ be a regular sequence in R and set $\overline{R} = R/(\boldsymbol{x})$. The canonical homomorphism $\psi\colon R \to \overline{R}$ induces isomorphisms

$$\pi_j(\psi)\colon \pi_j(R) \cong \pi_j(\overline{R}) \qquad \text{for} \quad j \geq 3$$

and an exact sequence

$$0 \to \pi_2(R) \xrightarrow{\pi_2(\psi)} \pi_2(\overline{R}) \to (\boldsymbol{x})/\mathfrak{m}(\boldsymbol{x}) \to \pi_1(R) \xrightarrow{\pi_1(\psi)} \pi_1(\overline{R}) \to 0\,.$$

In particular, one sees from here: if R is regular, then $\pi_i(R) = 0$ for $i \neq 1$; if R is a complete intersection, then $\pi_i(R) = 0$ for $i \neq 0, 1$.

5.9. **Lemma.** *Let $(R, \mathfrak{m}, k)$ be a local ring, $\mathfrak{c}$ an ideal contained in $\mathfrak{m}^2$, and $\boldsymbol{x}$ a regular sequence. Set $S = R/\mathfrak{c}$ and denote φ the canonical homomorphism $R \to S$. Also, set $\overline{R} = R/(\boldsymbol{x})$ and $\overline{S} = R/(\mathfrak{c}, \boldsymbol{x})$. If the induced homomorphism $\overline{\varphi}\colon \overline{R} \to \overline{S}$ is small, then φ is small.*

Proof. The naturality of the module of indecomposables yields a commutative diagram

$$\begin{array}{ccc}
\pi_*(R) & \xrightarrow{\pi_*(\varphi)} & \pi_*(S) \\
{\scriptstyle \pi_*(\psi)}\downarrow & & \downarrow \\
\pi_*(\overline{R}) & \xrightarrow{\pi_*(\overline{\varphi})} & \pi_*(\overline{S})
\end{array}$$

Since $\overline{\varphi}$ is small, the map $\pi_*(\overline{\varphi})$ is injective. Also, $\pi_{\geqslant 2}(\psi)$ is injective by 5.8, so the commutativity of the diagram implies that $\pi_{\geqslant 2}(\varphi)$ injective. Since $\mathfrak{c} \subseteq \mathfrak{m}^2$, the map $\pi_1(\varphi)$ is bijective by 5.7. Thus, $\pi_*(\varphi)$ is injective, hence φ is small by 5.6. □

Proof of Proposition 5.4. By 5.2 we can consider a minimal Cohen presentation $R = Q/\mathfrak{a}$. Also, the hypothesis $\mathfrak{m} \neq 0$ implies $\boldsymbol{A}(R) \geq 1$.

(1) By 5.7 we have

$$\inf\{i \geq 1 \mid \mathfrak{a} \cap \mathfrak{n}^{i+2} \subseteq \mathfrak{n}\mathfrak{a}\} = \inf\{i \geq 1 \mid \pi_2(\rho^{(i+1)}) \text{ is injective}\}\,.$$

By 5.6 the index $\boldsymbol{A}(R)$ is an upper bound for the number on the right hand side.

(2) Let $n \geq 1$ be an integer such that $\mathfrak{a} \cap \mathfrak{n}^{n+2} \subseteq \mathfrak{n}\mathfrak{a}$. Set $S = R/\mathfrak{m}^{n+1}$ and $\varphi = \rho^{(n+1)}$. Note that $S = Q/\mathfrak{b}$ with $\mathfrak{b} = (\mathfrak{a}, \mathfrak{n}^{n+1})$. We have $\mathfrak{b} \subseteq \mathfrak{n}^2$ and $\mathfrak{a} \cap \mathfrak{n}\mathfrak{b} \subseteq \mathfrak{n}\mathfrak{a}$, so $\pi_1(\varphi)$ and $\pi_2(\varphi)$ are injective by 5.7. Also, by 5.9 we have $\pi_i(R) = 0$ for $i > 2$, so $\pi_*(\varphi)$ is injective, and then φ is small by 5.6.

(3) We have $\boldsymbol{a} = (a)$ and the multiplicity of R is equal to the smallest integer n such that $a \in \mathfrak{n}^n$. The equality $\boldsymbol{A}(R) = \max\{1, \operatorname{mult}(R) - 1\}$ then follows from (2).

(4) The inequality $\boldsymbol{A}(R) \leq \boldsymbol{A}(R/(\boldsymbol{x}))$ follows by applying Lemma 5.9 to $\mathfrak{c} = \mathfrak{m}^{n+1}$, where $n = \boldsymbol{A}(R/(\boldsymbol{x}))$. Note that $n \geq 1$, cf. 5.3 □

5.10**. Remark.** The proof above shows that the number $\inf\{i \geq 1 \mid \mathfrak{a} \cap \mathfrak{n}^{i+2} \subseteq \mathfrak{n}\mathfrak{a}\}$ does not depend on the minimal Cohen presentation $R = Q/\mathfrak{a}$ and is an invariant of the ring R; we denote it $s(R)$.

5.11**. Remark.** If $n > \boldsymbol{A}(R)$, then Proposition 5.4(1) implies $\mathfrak{a} \cap \mathfrak{n}^{n+1} \subseteq \mathfrak{n}\mathfrak{a}$, that is, any minimal system of generators of $\mathfrak{a}$ is part of a minimal system of generators of $(\mathfrak{a}, \mathfrak{n}^n)$.

6. Golod rings and Golod homomorphisms

A surjective homomorphism $\varphi\colon R \to S$ is called *Golod* if

$$\mathrm{P}_k^S(t) = \frac{\mathrm{P}_k^R(t)}{1 - t\big(\mathrm{P}_S^R(t) - 1\big)}\,.$$

Levin [19, 3.15] proves that $\rho^{(n)}$ is Golod for all large n.

6.1**.** We define the *Golod index* of R by the formula

$$\boldsymbol{G}(R) = \inf\{s \geq 0 \mid \rho^{(n)} \text{ is Golod for all } n > s\}\,.$$

Note that $\boldsymbol{G}(R)$ is one less than the Golod invariant $G(R)$ introduced in [17].

6.2**.** By [3, 3.5] a Golod homomorphism is small. In particular, one has

$$\boldsymbol{A}(R) \leq \boldsymbol{G}(R)\,.$$

6.3**.** The condition $\boldsymbol{G}(R) = 0$ holds if and only if R is a field. Indeed, if $\boldsymbol{G}(R) = 0$, then 6.2 implies $\boldsymbol{A}(R) = 0$, hence R is a field by 5.3. The converse is clear.

Golod [12] studied local rings $(R, \mathfrak{m}, k)$ satisfying

$$\mathrm{P}_k^R(t) = \frac{(1+t)^{\operatorname{edim} R}}{1 - \sum_{j=1}^{\infty} \operatorname{rank} \mathrm{H}_j(K^R) t^{j+1}}\,,$$

Rings with this property are now called *Golod rings*. If $R = Q/\mathfrak{a}$ is a minimal Cohen presentation as in 1.7, then $\mathrm{H}(K^R) \cong \operatorname{Tor}_*^Q(R, k)$, hence the projection $Q \to Q/\mathfrak{a}$ is a Golod homomorphism if and only if R is a Golod ring.

6.4**.** Poincaré series are invariant under completion, so a surjective homomorphism $\varphi\colon R \to S$ is Golod if and only if the induced homomorphism $\widehat{\varphi}\colon \widehat{R} \to \widehat{S}$ is Golod. Thus, $\boldsymbol{G}(R) = \boldsymbol{G}(\widehat{R})$ and we may assume whenever necessary that R is complete.

Next we show that in some cases the indices $\boldsymbol{A}(R)$ and $\boldsymbol{G}(R)$ are equal.

6.5**. Theorem.** *An equality $\boldsymbol{A}(R) = \boldsymbol{G}(R)$ holds if one of the following conditions is satisfied:*

(1) *R is Golod and Artinian. In this case both indices are equal to* $\operatorname{pol\,reg}(R)$.
(2) $\operatorname{edim} R \le \dim R + 1$.
(3) $\operatorname{edim} R = 2$.
(4) $\operatorname{edim} R = 3$ *and R is a complete intersection.*

The proof of Theorem 6.5 requires some preparation. One of the ideas involved is to connect Golod rings to Golod homomorphisms. To do this we use their cohomological characterizations. As in Section 5, $\pi_*(R)$ denotes the module of indecomposables of Tor. Its vector space dual $\pi^*(R) = \operatorname{Hom}_k(\pi_*(R), k)$ is a graded Lie algebra, called the *homotopy Lie algebra* of R; we refer to [4, §10] for details.

6.6**.** Avramov and Löfwall prove that a local ring S is Golod if and only if $\pi^{\geqslant 2}(S)$ is a free Lie algebra and that a homomorphism $\varphi\colon R \to S$ is Golod if and only if the kernel of the induced map $\pi^*(\varphi)\colon \pi^*(S) \to \pi^*(R)$ is a free Lie algebra, cf. [3, 3.5], [5, 3.4], [22, Corollary 2.4].

6.7**. Proposition.** *Let $\varphi\colon R \to S$ be a surjective homomorphism of local rings. If φ is small and the ring S is Golod, then the homomorphism φ is Golod.*

Proof. The map $\pi^*(\varphi)$ is surjective by 5.6, and $\pi^1(\varphi)$ is bijective by 5.7. Denoting L the kernel of $\pi^*(\varphi)$, we have $L^1 = 0$. If S is Golod, then $\pi^{\geqslant 2}(S)$ is a free Lie algebra, cf. 6.6. Subalgebras of free Lie algebras are free by [18, A.1.10], hence $L = L^{\geqslant 2}$ is free and φ is Golod by 6.6. □

We recall two facts on Koszul complexes, using the notation of 3.5.

6.8**.** ([20, 1.6]) If there is an R-submodule V of $\mathfrak{m}K^R$ with $V^2 = 0$ and such that

$$\mathrm{Z}_{\ge 1}(K^R) \subseteq V + \mathrm{B}(K^R)\,,$$

then the ring R is Golod.

6.9**.** ([36, §2, Lemma 1]). If $(Q, \mathfrak{n}, k)$ is a regular local ring and p is a positive integer, then

$$\partial(\mathfrak{n}^{p-1} K^Q_{\geqslant 1}) = \partial(K^Q) \cap \mathfrak{n}^p K^Q\,.$$

In view of Proposition 6.7, we plan to prove most of Theorem 6.5 by showing that, under the given assumption, the ring $R/\mathfrak{m}^n$ is Golod for all $n > \boldsymbol{A}(R)$. If $\operatorname{edim} R = \dim R$, that is, if R is regular, then by Golod's example [12] the ring $R/\mathfrak{m}^n$ is Golod for all $n \ge 2$. The next proposition generalizes this result; the proof uses some ideas from of [37, §2, Lemma 2] and [35, §5, Lemma 2].

6.10**. Proposition.** *If $(R, \mathfrak{m}, k)$ is a local ring with* $\operatorname{edim} R \le \dim R + 1$*, then the ring $R/\mathfrak{m}^n$ is Golod for each integer $n \ge 2$.*

Proof. By 6.4 we may assume that R has a minimal Cohen presentation $R = Q/\mathfrak{a}$. Since $\operatorname{edim} R \le \dim R + 1$ and Q is catenary, it follows that $\operatorname{ht} \mathfrak{a} \le 1$. Since Q is factorial, there exist an element $x \in \mathfrak{n}$ and an ideal $\mathfrak{b}$ such that $\mathfrak{a} = x\mathfrak{b}$. Let s be the largest integer for which $x \in \mathfrak{n}^s$. We denote $\overline{R}$ the ring $R/\mathfrak{m}^n = Q/(\mathfrak{n}^n, x\mathfrak{b})$ and $\overline{\mathfrak{m}}$ its maximal ideal. Set $K = K^Q$ and $\overline{K} = K^R$. By 3.5 we have

$$\overline{K} = K \otimes_Q \overline{R} = K/(\mathfrak{n}^n, x\mathfrak{b})K\,.$$

If $\overline{y}$ is a cycle in $\overline{K}$ of degree $j \geq 1$, and y its preimage in K, then

$$\partial(y) = a + xb \quad \text{with} \quad a \in \mathfrak{n}^n K_{j-1} \quad \text{and} \quad b \in \mathfrak{b}K_{j-1}\,.$$

Differentiating, we obtain $x\partial(b) = -\partial(a) \in \mathfrak{n}^{n+1}K$. Now K is a complex of free Q-modules. If $c_1, \dots, c_r$ are the coefficients of $\partial(b)$ in a basis of K_{j-2}, then $xc_i \in \mathfrak{n}^{n+1}$ for all i. Since x is not contained in $\mathfrak{n}^{s+1}$ and Q is a regular ring, we conclude that $c_i \in \mathfrak{n}^{n+1-s}$, hence $\partial(b) \in \mathfrak{n}^{n+1-s}K$.

Let $T_1, \dots T_e$ be a basis of K_1, with $\partial(T_i) = g_i$ for each i. Note that $g_1, \dots, g_e$ minimally generate $\mathfrak{n}$, hence $x = a_1g_1 + \dots a_eg_e$ with $a_i \in \mathfrak{n}^{s-1}$. For $t = a_1T_1 + \dots + a_eT_e$ we have $t \in \mathfrak{n}^{s-1}K$ and $\partial(t) = x$. Setting $u = y - tb$, we then obtain

$$\partial(u) = a + xb - \partial(t)b + t\partial(b) = a + t\partial(b) \in \mathfrak{n}^n K\,.$$

By 6.9 there is an element $v \in \mathfrak{n}^{n-1}K$ such that $\partial(u) = \partial(v)$. Since u is a cycle of positive degree in K, we have $u - v = \partial(w)$ for some $w \in K$. In conclusion, any $\overline{y} \in \mathrm{Z}_{\geqslant 1}(\overline{K})$ can be written as

$$\overline{y} = \overline{t}\,\overline{b} + \overline{v} + \partial(w) \quad \text{with} \quad \overline{b} \in \overline{\mathfrak{b}} \quad \text{and} \quad \overline{v} \in \overline{\mathfrak{m}}^{n-1}\overline{K}$$

The submodule $V = (\overline{t}\,\overline{\mathfrak{b}}, \overline{\mathfrak{m}}^{n-1}\overline{K})$ is contained in $\overline{\mathfrak{m}}\overline{K}$. Indeed, if the ideal $\overline{\mathfrak{b}}$ is not contained in $\overline{\mathfrak{m}}$, then $s \geq 2$ and hence $\overline{t} \in \mathfrak{m}$. The product of any two cycles of the form $\overline{t}\,\overline{b} + \overline{v}$, with $\overline{b} \in \overline{\mathfrak{b}}$ and $\overline{v} \in \overline{\mathfrak{m}}^{n-1}\overline{K}$ is equal to zero, hence the ring R is Golod by 6.8. □

To continue, we need two more results of Avramov, Kustin and Miller [8].

6.11. ([8, 6.1]) Let S be homomorphic image of a regular local ring Q. If $\operatorname{pd}_Q(S) \leq 3$, then there is a Golod homomorphism from a complete intersection (of codimension less than 2) onto S.

6.12. ([8, 5.13]) Let $(R, \mathfrak{m}, k)$ be a local ring, $\mathfrak{c}$ an ideal contained in $\mathfrak{m}^2$, and set $\overline{R} = R/\mathfrak{c}$. If the natural homomorphism $R \to \overline{R}$ is Golod and $\mathbf{x} = x_1, \dots, x_r$ is a regular sequence that can be extended to a minimal generating set for $\mathfrak{c}$, then the induced homomorphism $R/(\boldsymbol{x}) \to \overline{R}$ is Golod.

Proof of Theorem 6.5. Let n be an integer such that $n > \boldsymbol{A}(R)$, that is, the map $\rho^{(n)}$ is small. We may assume $\boldsymbol{A}(R) > 0$, hence $n \geq 2$ (otherwise, R is a field and both indices are zero). We have to prove that $\rho^{(n)}$ is Golod.

(1) For an ideal $\mathfrak{c}$ of R the canonical map $R \to R/\mathfrak{c}$ is small if and only if $\mathfrak{c} = (0)$, cf. [3, 4.7]. We have thus $\mathfrak{m}^n = 0$, hence $\rho^{(n)}$ is Golod for trivial reasons. By 1.5, $\operatorname{pol reg}(R)$ is the largest integer s for which $\mathfrak{m}^s \neq 0$, hence $\boldsymbol{A}(R) = \boldsymbol{G}(R) = \operatorname{pol reg}(R)$.

For the rest of the proof we assume $R = Q/\mathfrak{a}$ with $(Q, \mathfrak{n}, k)$ regular and $\mathfrak{a} \subseteq \mathfrak{n}^2$, cf. 1.7.

(2) By Proposition 6.10, the ring $\overline{R} = R/\mathfrak{m}^s$ is Golod for each $s \geq 2$, so Proposition 6.7 implies that $\rho^{(n)}$ is Golod.

(3)By Scheja [33, Satz 9] the ring $\overline{R} = R/\mathfrak{m}^n = Q/(\mathfrak{a}, \mathfrak{n}^n)$ is either Golod or a complete intersection. If it is Golod, then $\rho^{(n)}$ is Golod by Proposition 6.7. Assume now that $\overline{R}$ is a complete intersection. By Remark 5.11, a minimal system of generators of $\mathfrak{a}$ can be completed to a minimal system of generators of $(\mathfrak{a}, \mathfrak{n}^n)$, hence $\mathfrak{a}$ is generated by a regular sequence. If $\dim R > 0$, then $\operatorname{codim} R \leq 1$, so the problem is settled by (2). If $\dim R = 0$, then $\mathfrak{a}$ is generated by a maximal regular sequence and thus $(\mathfrak{a}, \mathfrak{n}^n) = \mathfrak{a}$. Therefore $\mathfrak{m}^n = 0$, and $\rho^{(n)}$ is the identity map.

(4) We have $\mathfrak{a} = (\boldsymbol{x})$ for a regular sequence $\boldsymbol{x} = x_1, \dots, x_c$. By Remark 5.11, $\boldsymbol{x}$ is part of a minimal system of generators of $(\boldsymbol{x}, \mathfrak{n}^n)$. Set $S = R/\mathfrak{m}^n = Q/(\mathfrak{n}^n, \boldsymbol{x})$. If $c \leq 1$, then the assertion follows from (2). If $c \geq 2$, then by 6.11 there exists a regular sequence $\boldsymbol{x}'$ in $(\boldsymbol{x}') \subseteq (\boldsymbol{x}, \mathfrak{m}^n)$, of length at most two, such that the map $Q/(\boldsymbol{x}') \to S$ is Golod. Examining the proof of [8, 2.17], we see that we can modify $\boldsymbol{x}'$ to be either x_1, x_2 or x_1. The conclusion then follows from 6.12. □

Comments. We proved that $\boldsymbol{A}(R) = \boldsymbol{G}(R)$ for all complete intersections R with $\operatorname{edim} R \leq 3$. It would be interesting to see whether there are complete intersections of higher embedding dimension for which the equality does not hold. So far I did not find any example of a local ring with $\boldsymbol{A}(R) < \boldsymbol{G}(R)$.

7. The Levin index of a ring

7.1. We define the *Levin index* of the ring R by the formula

$$\boldsymbol{L}(R) = \boldsymbol{L}_R(\mathfrak{m})\,.$$

We set $\mu^{(n)} = \mu_R^{(n)} \colon \mathfrak{m}^n \to \mathfrak{m}^{n-1}$ and note that $\mu^{(n)} = \mu_{\mathfrak{m}}^{(n-1)}$, hence we have

$$\boldsymbol{L}(R) = \inf\{s \geq 1 \mid \operatorname{Tor}_*^R(\mu^{(n)}, k) = 0 \text{ for all } n > s\} = \max\{1, \boldsymbol{L}_R(R) - 1\}\,.$$

7.2. **Theorem.** *For a local ring R there are inequalities*

$$\boldsymbol{G}(R) \leq \boldsymbol{L}(R) \leq \max\{1, \operatorname{pol\,reg}(R)\}\,.$$

Proof. The proof of [19, 3.15] shows that if $\operatorname{Tor}_*^R(\mu^{(n)}, k) = 0$ for some $n \geq 2$, then $\rho^{(n)}$ is Golod; this proves the first inequality. The second one follows from Theorem 3.3. □

7.3. The proofs of [21, Theorem 2] and [20, 2.8] contain calculations of Poincaré series based on a choice of an integer $s \geq 2$ such that $\operatorname{Tor}_*^R(\mu^{(n)}, k) = 0$ for all $n \geq s$, hence Theorem 7.2 yields

$$\mathrm{P}_k^{R/\mathfrak{m}^n}(t) = \frac{\mathrm{P}_k^R(t)}{1 - t^2 \operatorname{Hilb}_{\mathfrak{m}^n}^R(-t)\, \mathrm{P}_k^R(t)}$$

for each $n > \max\{1, \operatorname{pol\,reg}(R)\}$.

The next proposition follows by combining 5.4(3) and 6.5(1) with 7.2.

7.4. **Proposition.** *Let $(R, \mathfrak{m})$ be a local ring which is not a field. If R is a hypersurface or a Golod Artinian ring, then*

$$\boldsymbol{A}(R) = \boldsymbol{G}(R) = \boldsymbol{L}(R) = \max\{1, \operatorname{pol\,reg}(R)\}. \qquad \square$$

We characterize next rings that satisfy $\boldsymbol{L}(R) = 1$. A similar result for graded algebras is mentioned by Roos [29, Remark 3.4].

7.5. **Proposition.** *A local ring R is Koszul if and only if $\boldsymbol{L}(R) = 1$. If R is Koszul, then $\operatorname{Ext}_R^*(k, k)$ is generated by its elements of degree* 1.

Proof. If R is Koszul, then 3.2 gives $\boldsymbol{L}(R) = \max\{1, \operatorname{reg}_R(R)\} = 1$. Conversely, assume that $\boldsymbol{L}(R) = 1$. We denote (X, ∂) a minimal free resolution of k over R and set $U = \mathrm{E}(X)$; this is the associated graded complex of X with respect to the natural filtration, as defined in 2.1 The fact that $\boldsymbol{L}(R) = 1$ means that the natural map $\mathrm{H}_*(\mathfrak{m}^n X) \to \mathrm{H}_*(\mathfrak{m}^{n-1} X)$ is zero for all $n \geq 1$. To prove that R is Koszul, we show that the complex U is acyclic. For all integers i and n the module $\mathrm{H}_i(U)_{i+n}$ is the homology of the complex

$$\mathfrak{m}^{n-1} X_{i+1}/\mathfrak{m}^n X_{i+1} \to \mathfrak{m}^n X_i/\mathfrak{m}^{n+1} X_i \to \mathfrak{m}^{n+1} X_{i-1}/\mathfrak{m}^{n+2} X_{i-1}\,.$$

Let $\overline{x}$ be an element in $\mathrm{Z}_i(U)_{i+n}$ for some $i \geq 1$, that is, $x \in \mathfrak{m}^n X_i$ and $\partial(x) \in \mathfrak{m}^{n+2} X_{i+1}$. Since the map $\mathrm{H}_*(\mathfrak{m}^{n+2} X) \to \mathrm{H}_*(\mathfrak{m}^{n+1} X)$ is zero, it follows that $\partial(x) = \partial(a)$ for some $a \in \mathfrak{m}^{n+1} X_i$. Thus $x - a$ is a cycle of X of positive degree, hence $x - a = \partial(b)$ for some $b \in X_{i+1}$. Since $\partial(b) \in \mathfrak{m}^n X$ and the map $\mathrm{H}_*(\mathfrak{m}^n X) \to \mathrm{H}_*(\mathfrak{m}^{n-1} X)$ is zero, it follows that $\partial(b) = \partial(c)$ for some $c \in \mathfrak{m}^{n-1} X_{i+1}$. We conclude that $\overline{x} = \partial(\overline{c})$ in U, hence U is acyclic and thus $\operatorname{reg}_R(k) = 0$ by 2.3.

The last assertion follows by 5.3. □

For graded k-algebras, the converse of the last assertion of the proposition holds by [22, Theorem 1.2]. However, the converse does not hold for local rings, as can be seen from the following example:

7.6. Let $(Q, \mathfrak{n}, k)$ be a 2-dimensional regular local ring and u, v a system of parameters. We set $\mathfrak{a} = (u^2 + v^3, uv)$ and $R = Q/\mathfrak{a} = k[u,v]/(u^2 + v^3, uv)$. This is a local complete intersection such that $\operatorname{Ext}^*_R(k,k)$ is generated in degree 1, cf. [37, Theorem 5]. Still, $\operatorname{gr}(R) \cong k[u,v]/(u^2, uv, v^4)$ and $\beta^{\operatorname{gr}(R)}_{2,4} \neq 0$, hence $\operatorname{gr}(R)$, and thus R is not Koszul.

8. Reduction by a regular sequence

In this section we study the behavior of the Levin index under factorization of a regular sequence, in connection with a question of Roos. In [29] and [30] he introduces the graded vector spaces

$$\overline{\mathrm{S}}_{\mathfrak{m}^n} = \operatorname{Im}\left(\operatorname{Ext}^{\geqslant 1}_R(\psi^{(n)}, k)\right) \subseteq \operatorname{Ext}^{\geqslant 1}_R(R/\mathfrak{m}^{n+1}, k)\,,$$

where $\psi^{(n)}\colon R/\mathfrak{m}^{n+1} \to R/\mathfrak{m}^n$ is the canonical map, and considers the following properties of the ring R:

($\mathcal{M}_s$) $\overline{\mathrm{S}}_{\mathfrak{m}^n} = 0$ for all $n > s$.

($\mathcal{L}_s$) R satisfies $\mathcal{M}_s$ and the induced homomorphisms $\overline{\mathrm{S}}_{\mathfrak{m}^n} \to \overline{\mathrm{S}}_{\mathfrak{m}^{n+1}}$ are bijective for $n = 1, \dots, s-3$.

As pointed out in [29, Remark 3.4], a graded k-algebra R is Koszul if and only if it satisfies $\mathcal{M}_2$. In general, these properties measure how far the algebra is from being Koszul.

8.1. Roos [29, 7(iv)] asks the following question: Let R be a Cohen-Macaulay ring and $\overline{R} = R/(\text{an } R\text{-sequence})$. Is any of the conditions $\mathcal{L}_s$ and/or $\mathcal{M}_s$ true for R if and only if it is true for $\overline{R}$? Although in [29] the ring R is graded, the question makes sense for local rings as well.

The isomorphism of functors $\operatorname{Hom}_k\left(\operatorname{Tor}^R_{\geqslant 1}(-,k), k\right) \cong \operatorname{Ext}^{\geqslant 1}_R(-,k)$ shows that ($\mathcal{M}_s$) is equivalent to $\boldsymbol{L}(R) < s$. Thus, the question of Roos can be partly reformulated as whether the Levin index is invariant under factorization of a regular sequence. We show next that the answer is negative, unless certain assumptions are made on the regular sequence.

8.2. Let $(R,\mathfrak{m})$ be a regular local ring with $\dim(R) > 0$. If x is an element in $\mathfrak{m}^n \smallsetminus \mathfrak{m}^{n+1}$ for some integer $n \geq 1$, then $\boldsymbol{L}(R) = 1$ and $\boldsymbol{L}(R/(x)) = n-1$ by 7.4 and 1.9.

A sequence $\boldsymbol{x} = x_1,\dots,x_m$ of elements in R is called *strictly regular* if the initial forms $x_1^*,\dots,x_m^*$ form a regular sequence. It is known and easy to see that a strictly regular sequence is an R-sequence.

We give next a partial answer to Roos's question.

8.3. **Theorem.** *If $(R,\mathfrak{m})$ is a local ring and $\boldsymbol{x} = x_1,\dots,x_m$ is a strictly regular sequence in $\mathfrak{m} \smallsetminus \mathfrak{m}^2$, then the following hold:*

(1) $\boldsymbol{L}(R) = \boldsymbol{L}\left(R/(\boldsymbol{x})\right)$.
(2) *R satisfies $\mathcal{M}_s$ if and only if $R/(\boldsymbol{x})$ satisfies $\mathcal{M}_s$.*
(3) *R satisfies $\mathcal{L}_s$ if and only if $R/(\boldsymbol{x})$ satisfies $\mathcal{L}_s$.*

The proof of the theorem follows from Proposition 8.7 below. Here are some preliminaries:

8.4. ([6, 3.1]) Let X be a minimal Tate resolution of k over R (see 5.5) and x a regular element in $\mathfrak{m} \smallsetminus \mathfrak{m}^2$. If $T \in X_1$ satisfies $\partial(T) = x$, then $X/(x,T)X$ is a minimal free resolution of k over $R/(x)$.

8.5. **Lemma.** *Let $(R,\mathfrak{m},k)$ be a local ring, x be an element of R, and T be an element of X_1 with $\partial(T) = x$. If $(\mathfrak{m}^{n+1} \colon x) \subseteq \mathfrak{m}^n$ and $\partial(Tc) \in \mathfrak{m}^{n+1}X$ for some integer n and some $c \in X$, then $Tc \in \mathfrak{m}^n X$.*

Proof. The Leibnitz rule gives $\partial(Tc) = xc - T\partial(c)$. Multiplying by T, we obtain $T\partial(Tc) = T(xc) = x(Tc)$. This implies $x(Tc) \in \mathfrak{m}^{n+1}X$. We set $b = Tc \in X_s$ (where s is the homological degree). Since X_s is a free R-module, we consider $b_1,\dots,b_r$ to be the components of b in a basis. Then $b_i x \in \mathfrak{m}^{n+1}$ for any i. The assumption on x implies $b_i \in \mathfrak{m}^n$, hence $Tc \in \mathfrak{m}^n X$. □

8.6. If k is infinite, then each superficial element x which is regular satisfies the condition $(\mathfrak{m}^{n+1} \colon x) \subseteq \mathfrak{m}^n$ for all large integers n, cf. [31, Remarks, I-9]; such elements exist by [31, 3.2, I-8]. Also, if $x \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ is strictly regular, then $(\mathfrak{m}^{n+1} \colon x) \subseteq \mathfrak{m}^n$ for all $n \geq 0$.

8.7. **Proposition.** *Let $(R,\mathfrak{m},k)$ be a graded Noetherian ring, x a regular element not contained in $\mathfrak{m}^2$, and set $\overline{R} = R/(x)$ and $\overline{\mathfrak{m}} = \mathfrak{m}/(x)$. For any positive integer n such that $(\mathfrak{m}^{n+1} \colon x) \subseteq \mathfrak{m}^n$ the following hold:*

(1) *The induced homomorphism $\overline{\mathrm{S}}_{\overline{\mathfrak{m}}^n} \to \overline{\mathrm{S}}_{\mathfrak{m}^n}$ is surjective.*
(2) *The induced homomorphism $\overline{\mathrm{S}}_{\overline{\mathfrak{m}}^{n-1}} \to \overline{\mathrm{S}}_{\mathfrak{m}^{n-1}}$ is injective.*

Proof. We first reformulate the statement in terms of homology. For each s we denote $\mathrm{U}_{\mathfrak{m}^s}$ the image of $\left(\mathrm{Tor}^R_{\geqslant 1}(\psi^{(s)},k)\right)$ in $\mathrm{Tor}^R_{\geqslant 1}(R/\mathfrak{m}^s,k)$. In view of the isomorphisms $\mathrm{Hom}_k\left(\mathrm{Tor}^R_{\geqslant 1}(-,k),k\right) \cong \mathrm{Ext}_R^{\geqslant 1}(-,k)$, we conclude that $\mathrm{U}_{\mathfrak{m}^s}$ is canonically isomorphic to the vector-space dual of $\overline{\mathrm{S}}_{\mathfrak{m}^s}$. Thus, for (1) we have to prove that the induced map $\mathrm{U}_{\mathfrak{m}^n} \to \mathrm{U}_{\overline{\mathfrak{m}}^n}$ is injective and for (2) we have to prove that the induced map $\mathrm{U}_{\mathfrak{m}^{n-1}} \to \mathrm{U}_{\overline{\mathfrak{m}}^{n-1}}$ is surjective.

Let X be a minimal Tate resolution of k over R. By 8.4 the complex $X/(x,T)X$ is a minimal free resolution of k over R/x, where $T \in X_1$ satisfies $\partial(T) = x$. For all integers i we identify $\mathrm{Tor}^R_{\geqslant 1}(R/\mathfrak{m}^i,k)$ with $\mathrm{H}_{\geqslant 1}(X/\mathfrak{m}^i X)$ and $\mathrm{Tor}^{\overline{R}}_{\geqslant 1}(\overline{R}/\overline{\mathfrak{m}}^i,k)$

with $\mathrm{H}_{\geqslant 1}\big(X/(\mathfrak{m}^i, x, T)X\big)$. Overbars denote residue classes, as appropriate to the context.

(1) Let $\mathrm{cls}(\overline{y})$ be an element of $\mathrm{H}_{\geqslant 1}(X/\mathfrak{m}^{n+1}X)$ whose image in $\mathrm{U}_{\overline{\mathfrak{m}}^n}$ zero. Thus $\partial(y) \in \mathfrak{m}^{n+1}X$ and

$$y = \partial(a) + b + xc + Tg \qquad \text{with} \quad b \in \mathfrak{m}^n X \quad \text{and} \quad a, c, g \in X\,.$$

The Leibnitz rule gives $xc = \partial(Tc) + T\partial(c)$ and we obtain:

$$y = \partial(a) + b + \partial(Tc) + T\partial(c) + Tg = \partial(a') + b + Tg' \qquad \text{with} \quad a', g' \in X\,.$$

Differentiating, we get $\partial(Tg') = \partial(y) - \partial(b)$. Since $\partial(y) \in \mathfrak{m}^{n+1}X$ and $b \in \mathfrak{m}^n X$, we obtain $\partial(Tg') \in \mathfrak{m}^{n+1}X$. By Lemma 8.5 we have $Tg' \in \mathfrak{m}^n X$ and thus $\overline{y}$ is a boundary in $X/\mathfrak{m}^n X$. We thus have $\mathrm{cls}(\overline{y}) = 0$ in $\mathrm{H}_{\geqslant 1}(X/\mathfrak{m}^n X)$, hence the map $\mathrm{U}_{\mathfrak{m}^n} \to \mathrm{U}_{\overline{\mathfrak{m}}^n}$ is injective

(2) For $\mathrm{cls}(\overline{y}) \in \mathrm{H}_{\geqslant 1}\big(X/(\mathfrak{m}^n, x, T)X\big)$ we have

$$\partial(y) = a + xb + Tc \qquad \text{with} \quad a \in \mathfrak{m}^n X \quad \text{and} \quad b, c \in X\,.$$

The Leibnitz rule gives $xb = \partial(Tb) + T\partial(b)$ and thus

$$\partial(y - Tb) = a + Tc' \qquad \text{with} \quad c' \in X\,.$$

Differentiating, we obtain $\partial(Tc') = -\partial(a)$; since $a \in \mathfrak{m}^n X$, we have $\partial(a) \in \mathfrak{m}^{n+1}X$. Lemma 8.5 yields $Tc' \in \mathfrak{m}^n X$, hence $\partial(y - Tb) \in \mathfrak{m}^n X$. We conclude that $\overline{y} - \overline{T}\,\overline{b}$ is a cycle in $X/\mathfrak{m}^n X$ and thus $\mathrm{cls}(\overline{y})$ is the image of $\mathrm{cls}(\overline{y} - \overline{T}\,\overline{b}) \in \mathrm{H}_{\geqslant 1}(X/\mathfrak{m}^n X)$. Thus, the induced map $\mathrm{H}_{\geqslant 1}(X/\mathfrak{m}^n X) \to \mathrm{H}_{\geqslant 1}\big(X/(\mathfrak{m}^n, x, T)X\big)$ is surjective. This implies the surjectivity of the map $\mathrm{U}_{\mathfrak{m}^{n-1}} \to \mathrm{U}_{\overline{\mathfrak{m}}^{n-1}}$. □

9. Graded rings

In this section we consider graded Noetherian rings. Let k be a field. Adapting the notation of a local ring, we denote $(R, \mathfrak{m}, k)$ a graded Noetherian ring $R = \oplus_{i=0}^{\infty} R_i$ satisfying $R = R_0[R_1]$, with maximal irrelevant ideal $\mathfrak{m} = \oplus_{i=1}^{\infty} R_i$ and $R_0 = k$. We use the notation k also for the residue field $R/\mathfrak{m}$. All R-homomorphisms are assumed homogeneous. The notions and results used so far have analogous graded versions. We only mention that the notion corresponding to a minimal Cohen presentation is a presentation of the form $R = k[u_1, \dots, u_r]/\mathfrak{a}$, with a homogeneous ideal $\mathfrak{a} \subseteq \mathfrak{n}^2$, where $\mathfrak{n} = (u_1, \dots, u_r)$, and that the Koszul complexes K^R are understood as $K^R = K(\boldsymbol{g}, R)$ for a chosen basis $\boldsymbol{g}$ of R_1. Also, Tate resolutions become graded resolutions in a natural way.

9.1. Let $R = k[u_1, \dots, u_r]/\mathfrak{a}$ be a minimal presentation as above. We noted in Remark 5.10 that the number $s(R) = \inf\{i \geq 1 \mid \mathfrak{a} \cap \mathfrak{n}^{i+2} \subseteq \mathfrak{n}\mathfrak{a}\}$ does not depend on the choice of the presentation. It is easy to see that $s(R)$ is equal to 1 if $\mathfrak{a} = (0)$ and is one less the maximum of the degrees of a minimal system of generators of $\mathfrak{a}$, otherwise.

All the results of this paper have analogous versions for graded rings. There are also some improvements of the statements, which are collected in the next theorem.

9.2. **Theorem.** *Let R be a graded ring as above, which is not a field.*

(1) *The ring R is Koszul if and only if $\boldsymbol{A}(R) = 1$, if and only if $\boldsymbol{G}(R) = 1$, if and only if $\boldsymbol{L}(R) = 1$.*
(2) $s(R) \leq \boldsymbol{A}(R) \leq \boldsymbol{L}(R) \leq \max\{1, \operatorname{pol\,reg}(R)\}$.
(3) *If R is a complete intersection, then $\boldsymbol{A}(R) = s(R)$.*
(4) *If a linear form y is a non-zero divisor, then $\boldsymbol{L}(R) = \boldsymbol{L}(R/(y))$.*

(5) *If R is Golod, then* $\boldsymbol{A}(R)=\boldsymbol{G}(R)=\boldsymbol{L}(R)=\max\{1,\operatorname{pol\,reg}(R)\}$.

The proofs of (1)–(4) are mainly contained in the previous sections. The only part that needs a proof is (5).

9.3. We denote $\operatorname{MH}(K^R)$ the set of all matric Massey products of $\operatorname{H}_{\geqslant 1}(K^R)$, as defined in ([23, §1]. It is a submodule of $\operatorname{H}_{\geqslant 1}(K^R)$, containing the usual products. The map $\operatorname{H}(K^\varphi)\colon K^R\to K^S$ induced by a homomorphism of local rings $\varphi\colon R\to S$ satisfies $\operatorname{H}(K^\varphi)(\operatorname{MH}(K^R))\subseteq \operatorname{MH}(K^S)$ (see [23, 3.10]). By [3, 4.6], if φ is small then the induced homomorphism

$$\widetilde{\varphi}\colon\ \operatorname{H}_{\geqslant 1}(K^R)/\operatorname{MH}(K^R)\to \operatorname{H}_{\geqslant 1}(K^S)/\operatorname{MH}(K^S)$$

is injective. Also, Golod [12] shows that the ring R is Golod if and only if $\operatorname{MH}(K^R)=0$, cf. [5, (2.3)]. For our purposes, we will use the graded version of these results.

Proof of Proposition 9.2(5). Set $n=\boldsymbol{A}(R)+1$ and $S=R/\mathfrak{m}^n$. In particular, the map $\rho^{(n)}\colon R\to S$ is small. Note that $n\geq 2$ by 5.3. Since R is a Golod ring, we have $\operatorname{MH}(K^R)=0$ by 9.3 and then the induced map $\operatorname{H}_{\geqslant 1}(K^R)\to \operatorname{H}_{\geqslant 1}(K^S)$ is injective by 9.3. Set $s=\operatorname{pol\,reg}(R)$. There exists then an integer i such that $\operatorname{H}_i(K^R)_{i+s}\neq 0$, hence $\operatorname{H}_i(K^S)_{i+s}\neq 0$. Since $K^S=K^R/\mathfrak{m}^nK^R$ by 3.5, we observe that $\operatorname{H}_i(K^S)_j=0$ for all $j\geq n+i$. We conclude that $i+s<n+i$, that is $n>s$, hence $\boldsymbol{A}(R)\geq s$. From Theorem 9.2(2) we also know that $\boldsymbol{L}(R)\leq\max\{1,s\}$, so the inequalities between the indices give the desired equalities. □

Comments. By 6.5 and 9.2(5) all Golod rings R which are either hypersurfaces, or local and Artinian, or graded satisfy the equality $\boldsymbol{A}(R)=\boldsymbol{G}(R)$. I do not know whether Golod local rings of positive dimension satisfy this equality.

9.4. If R is a complete intersection on quadrics, then R is a Koszul algebra. Thus $s(R)=\boldsymbol{A}(R)=\boldsymbol{G}(R)=\boldsymbol{L}(R)=1$, while $\operatorname{pol\,reg}(R)=\operatorname{codim}R-1$. We note that all Koszul algebras which are not regular rings satisfy $s(R)=\boldsymbol{A}(R)=1$; thus, the equality between these two invariants is not specific to complete intersections.

9.5. Consider the ring $R=k[u,v]/(u^3,uv^2)$. Since $\operatorname{codim}(R)=1$, this ring is Golod by a result of Shamash [35, §5, Corollary (2)]. Note that $s(R)=2$ and $\operatorname{pol\,reg}(R)=3$. By Theorem 9.2(5) we have $\boldsymbol{A}(R)=\boldsymbol{G}(R)=\boldsymbol{L}(R)=3$. Thus, we have $\boldsymbol{A}(R)>s(R)$ in this case.

9.6. Consider the ring $R=k[u,v]/(u^3,v^3)$. Since $\operatorname{edim}(R)\leq 2$, we know that $2=s(R)=\boldsymbol{A}(R)=\boldsymbol{G}(R)$ by 6.5. Also, $\operatorname{pol\,reg}(R)=4$ and we can see that $\boldsymbol{L}(R)=4$. Indeed, by Tate [38, Theorem 4] a minimal free resolution of k over R has the form

$$X=R\langle S,Y,U,V\,|\,\partial(U)=u,\partial(V)=v,\partial(S)=u^2U,\partial(T)=v^2V\rangle\ .$$

Express $\operatorname{Tor}^R_{\geqslant 1}(R/\mathfrak{m}^4,k)$ as $\operatorname{H}_{\geqslant 1}(X/\mathfrak{m}^4X)$ and $\operatorname{Tor}^R_{\geqslant 1}(R/\mathfrak{m}^3,k)$ as $\operatorname{H}_{\geqslant 1}(X/\mathfrak{m}^3X)$. For degree reasons, the image of u^2UT is a cycle in $X/\mathfrak{m}^4X$, which is not a boundary in $X/\mathfrak{m}^3X$, hence the map $\operatorname{Tor}^R_{\geqslant 1}(R/\mathfrak{m}^4,k)\to\operatorname{Tor}^R_{\geqslant 1}(R/\mathfrak{m}^3,k)$ is not zero.

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