INDEPENDENCE OF THE TOTAL REFLEXIVITY CONDITIONS FOR MODULES

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Abstract. We show that the conditions defining total reflexivity for modules are independent. In particular, we construct a commutative Noetherian local ring R and a reflexive R-module M such that $\mathrm{Ext}^i_R(M,R)=0$ for all $i>0$, but $\mathrm{Ext}^i_R(M^*,R)\neq 0$ for all $i>0$.

INTRODUCTION

Let R be a commutative Noetherian ring. For any R-module M we set $M^* = \mathrm{Hom}_R(M,R)$. An R-module M is said to be *reflexive* if it is finite and the canonical map $M \to M^{**}$ is bijective. A finite R-module M is said to be *totally reflexive* if it satisfies the following conditions:

(i) M is reflexive
(ii) $\mathrm{Ext}^i_R(M,R)=0$ for all $i>0$
(iii) $\mathrm{Ext}^i_R(M^*,R)=0$ for all $i>0$.

This notion is due to Auslander and Bridger [1]: the totally reflexive modules are precisely the modules of *G-dimension* zero. The G-dimension of a module is one of the best studied non-classical homological dimensions, and is defined in terms of the length of a resolution of the module by totally reflexive modules.

Given any homological dimension, a serious concern is whether its defining conditions can be verified effectively. For example, the projective dimension of a finite R-module M is zero if and only if $\mathrm{Ext}^1_R(M,N)=0$ for all finite R-modules N. Furthermore, when R is local with maximal ideal $\mathfrak{m}$, one only needs to check vanishing for $N=R/\mathfrak{m}$. In the same spirit, it is natural to ask whether the set of conditions defining total reflexivity is overdetermined (cf. [4, §2]) and in particular, whether total reflexivity for a module can be established by verifying vanishing of only finitely many Ext modules.

When R is a local Gorenstein ring, (ii) implies the other two conditions above, and it is equivalent to M being maximal Cohen-Macaulay. Recently, Yoshino [9] studied other situations when (ii) alone implies total reflexivity, and raised the question whether this is always the case.

In the present paper we give an example of a local Artinian ring R which admits modules whose total reflexivity conditions are independent, in that (ii) implies neither (i) nor (iii); (i) and (ii) do not imply (iii), equivalently, (i) and (iii) do not imply (ii). More precisely, we prove the following result as Theorem 1.7:

Theorem. *There exists a local Artinian ring R, and a family $\{M_s\}_{s\geqslant 1}$ of reflexive R-modules such that*

(1) $\mathrm{Ext}^i_R(M_s,R)=0$ *if and only if* $1\le i\le s-1$;

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(2) $\operatorname{Ext}^i_R(M_s^*, R) = 0$ *for all* $i > 0$.

Moreover, there exists a non-reflexive R*-module* L *such that*

(1′) $\operatorname{Ext}^i_R(L, R) = 0$ *for all* $i > 0$*;*

(2′) $\operatorname{Ext}^i_R(L^*, R) \neq 0$ *for all* $i > 0$.

By taking $N_s = M_s^*$ for all $s \geq 1$, we get a statement dual to that of the first part above: there exits a family $\{N_s\}_{s\geqslant 1}$ of reflexive R-modules such that (1) $\operatorname{Ext}^i_R(N_s^*, R) = 0$ if and only if $1 \leq i \leq s-1$; (2) $\operatorname{Ext}^i_R(N_s, R) = 0$ for all $i > 0$.

This theorem shows that in order to check whether or not a module M is totally reflexive — even for a local Artinian ring — one needs to check vanishing of $\operatorname{Ext}^i_R(M, R)$ and $\operatorname{Ext}^i_R(M^*, R)$ for infinitely many values of i. In Section 2, however, we point out that when R is Artinian and standard graded, in the sense that $R = \bigoplus_{i=0}^{\infty} R_i$ with $R_0 = k$, a field, and $R = R_0[R_1]$, one may skip checking finitely many values of i of the same parity.

In our example, R is a standard graded Koszul algebra and has Hilbert series $\sum_{i\geq 0} \operatorname{rank}_k(R_i)t^i = 1 + 4t + 3t^2$. The ring R is thus local, and its maximal ideal $\mathfrak{m}$ satisfies $\mathfrak{m}^3 = 0$. Note that our example is minimal in the following sense: if $\mathfrak{m}^2 = 0$, then any finite R-module M which satisfies $\operatorname{Ext}^i_R(M, R) = 0$ for some $i > 1$ is totally reflexive, hence (ii) alone implies total reflexivity. (See 1.1 below.)

Our construction involves a minimal acyclic complex C of finite free R-modules such that the sequence $\{\operatorname{rank}_R(C_i)\}_{i\geqslant 0}$ is strictly increasing and has exponential growth, while the sequence $\{\operatorname{rank}_R(C_{-i})\}_{i\geqslant 0}$ is constant. In the last section we raise several related questions.

1. Independence

Let R be a Noetherian commutative ring and M a finite R-module. Suppose that $\phi\colon G \to F$ is a homomorphism of finite free R-modules with $M = \operatorname{Coker}\phi$. The R-module $\operatorname{Tr}(M) := \operatorname{Coker}\phi^*$ is called the *transpose* of M, and it is unique up to projective equivalence; it is thus well-defined in the stable category of R. The R-modules $\operatorname{Tr}(\operatorname{Tr}(M))$ and M are isomorphic up to projective summands and there is an exact sequence

$$0 \to \operatorname{Ext}^1_R(\operatorname{Tr}(M), R) \to M \to M^{**} \to \operatorname{Ext}^2_R(\operatorname{Tr}(M), R) \to 0$$

where the map in the middle is the natural map. In particular, M is reflexive if and only if $\operatorname{Ext}^i_R(\operatorname{Tr}(M), R) = 0$ for $i = 1, 2$. Also note that M^* is a second syzygy of $\operatorname{Tr}(M)$, hence $\operatorname{Ext}^i_R(M^*, R) \cong \operatorname{Ext}^{i+2}_R(\operatorname{Tr}(M), R)$ for all $i > 0$. Thus the definition of M being totally reflexive can be recast as follows:

$$\operatorname{Ext}^i_R(M, R) = 0 = \operatorname{Ext}^i_R(\operatorname{Tr}(M), R) \quad \text{for all} \quad i > 0\,.$$

It is convenient to have a uniform notation for the conditions above: let $i \in \mathbb{Z}$, $i \neq 0$. We say that M satisfies condition (TR_i) provided

$$(\mathsf{TR}_i): \qquad \begin{cases} \operatorname{Ext}^i_R(M, R) = 0 & \text{if } i \geq 1 \\ \operatorname{Ext}^{-i}_R(\operatorname{Tr}(M), R) = 0 & \text{if } i \leq -1. \end{cases}$$

Thus M is totally reflexive if and only if M satisfies (TR_i) for all $i \neq 0$.

Note that we did not define a condition (TR_i) for $i = 0$. When referring to these conditions we will assume tacitly that $i \neq 0$.

1.1. *Remarks.* (1) Assume that R is local Gorenstein. The module M is then totally reflexive if and only if (TR_i) is satisfied for all i with $0 < i \leq \dim R$.

(2) Assume that R is local, with maximal ideal $\mathfrak{m}$ and residue field k. If $\mathfrak{m}^2 = 0$, then the first syzygy N in a minimal free resolution of M has $\mathfrak{m}N = 0$, hence it is a finite dimensional k-vector space. If $\operatorname{Ext}^i_R(M,R) = 0$ for some $i > 1$ then either M is free or $\operatorname{Ext}^{i-1}_R(k,R) = 0$. Thus, when $\mathfrak{m}^2 = 0$ we obtain: M is totally reflexive whenever (TR_i) is satisfied for a single value of i with $i \neq \pm 1$.

(3) Assume that R is a commutative Noetherian ring. Yoshino proved in [9] that if the full subcategory of R-modules N with $\operatorname{Ext}^i_R(N,R) = 0$ for all $i > 0$ is of finite type, then every module M in this subcategory is totally reflexive.

The remarks above lead to the question: How many conditions (TR_i) does one need to check for total reflexivity, without placing extra assumptions on the ring? Is it possible that only finitely many suffice? Is it enough to check (TR_i) for all $i > 0$, or, more generally, for all $i > s$ for some integer s?

Theorem 1.7 (stated also in the introduction) provides negative answers to these questions: in general, one needs to check the conditions (TR_i) for infinitely many positive values of i and infinitely many negative values of i.

We now describe the ring of Theorem 1.7. Related rings were used by Gasharov and Peeva in [6] and then by the authors in [7] to disprove various conjectures.

Let k be a field which is not algebraic over a finite field and let $\alpha \in k$ be an element of infinite multiplicative order. For the remainder of this section we assume the ring R to be defined as follows.

1.2. Consider the polynomial ring $k[V,X,Y,Z]$ in four variables (each of degree one) and set

$$R = k[V,X,Y,Z]/I,$$

where I is the ideal generated by the following quadratic relations:

$$V^2,\ Z^2,\ XY,\ VX + \alpha XZ,\ VY + YZ,\ VX + Y^2,\ VY - X^2\,.$$

As a vector space over k, it has a basis consisting of the following 8 elements:

$$1,\ v,\ x,\ y,\ z,\ vx,\ vy,\ vz,$$

where v,x,y,z denote the residue classes of the variables modulo I. In particular, R has Hilbert series $H_R(t) = 1 + 4t + 3t^2$.

1.3. *Remark.* One may check that the generators for I listed above form a Gröbner basis for I. Therefore by [5, Section 4], the ring R is Koszul, and it follows that the Poincaré series $\sum_i \operatorname{rank}_k \operatorname{Tor}^R_i(k,k)t^i$ of the residue field is equal to $(1-4t+3t^2)^{-1}$.

For each integer $i \leq 0$ we let $d_i\colon R^2 \to R^2$ denote the map given with respect to the standard basis of R^2 by the matrix

$$\begin{pmatrix} v & \alpha^{-i}x \\ y & z \end{pmatrix}.$$

Also, let $d_1\colon R^3 \to R^2$ denote the map represented by the matrix

$$\begin{pmatrix} v & \alpha^{-1}x & yz \\ y & z & 0 \end{pmatrix}$$

and let $d_2\colon R^7 \to R^3$ be represented by the matrix

$$\begin{pmatrix} v & \alpha^{-2}x & -y & 0 & 0 & 0 & 0 \\ y & z & \alpha x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v & x & y & z \end{pmatrix}.$$

Consider a minimal free resolution of $\operatorname{Coker} d_2$ with d_2 as the first differential:

$$\cdots \to R^{b_i} \xrightarrow{d_i} R^{b_{i-1}} \to \cdots \to R^{b_3} \xrightarrow{d_3} R^7 \xrightarrow{d_2} R^3,$$

where for each $i \geq 3$ the map $d_i\colon R^{b_i} \to R^{b_{i-1}}$ denotes the $(i-1)$st differential in this resolution.

1.4. **Lemma.** *The sequence of homomorphisms:*

$$C: \quad \cdots \to R^{b_3} \xrightarrow{d_3} R^7 \xrightarrow{d_2} R^3 \xrightarrow{d_1} R^2 \xrightarrow{d_0} R^2 \xrightarrow{d_{-1}} R^2 \xrightarrow{d_{-2}} R^2 \xrightarrow{d_{-3}} R^2 \to \cdots$$

is a doubly infinite complex with $H_i(C) = 0$ for all $i \in \mathbb{Z}$.

Proof. (1). The defining equations of R guarantee that $d_{i-1}d_i = 0$ for all $i \leq 2$. For $i \geq 2$ the maps d_i are differentials in a free resolution, hence the equality holds for all $i \geq 3$, as well. We conclude that C is a complex and $H_i(C) = 0$ for all $i \geq 2$.

We let (a, b) denote an element of R^2 written in the standard basis of R^2 as a free R-module. For each $i \leq 0$ the k-vector space $\operatorname{Im} d_i$ is generated by the elements:

$$\begin{aligned} d_i(1,0) &= (v, y) & d_i(z,0) &= (vz, -vy) \\ d_i(0,1) &= (\alpha^{-i}x, z) & d_i(0,v) &= (\alpha^{-i}vx, vz) \\ d_i(v,0) &= (0, vy) & d_i(0,x) &= (\alpha^{-i}vy, -\alpha^{-1}vx) \\ d_i(x,0) &= (vx, 0) & d_i(0,y) &= (0, -vy) \\ d_i(y,0) &= (vy, -vx) & d_i(0,z) &= (-\alpha^{-i-1}vx, 0) \end{aligned}$$

It is clear that $\operatorname{rank}_k(\operatorname{Im} d_i) = 8$ for all $i \leq 0$. Since $\operatorname{rank}_k(R^2) = 16$, this implies that $\operatorname{rank}_k(\operatorname{Ker} d_i) = 8$ for all $i \leq 0$, showing that $H_i(C) = 0$ for all $i \leq -1$.

Notice that for $i = 1$ the elements above give 7 linearly independent elements in $\operatorname{Im} d_1$, and the 8th can be taken to be $\varepsilon(0,0,1) = (yz, 0)$. (Here (a, b, c) denotes an element of R^3 in its standard basis as a free R-module.) Thus $\operatorname{rank}_k(\operatorname{Im} d_1) \geq 8$, and so $\operatorname{rank}_k(\operatorname{Ker} d_1) \leq 16$. In particular, we obtain $H_0(C) = 0$.

To prove $H_1(C) = 0$ we need to show that $\operatorname{rank}_k(\operatorname{Im} d_2) \geq 16$. Indeed, the following elements in $\operatorname{Im} d_2$ are linearly independent:

$$\begin{aligned} d_2(e_1) &= (v, y, 0) & d_2(ye_4) &= (0, 0, vy) \\ d_2(e_2) &= (\alpha^{-2}x, z, 0) & d_2(ze_4) &= (0, 0, vz) \\ d_2(e_3) &= (-y, \alpha x, 0) & d_2(xe_1) &= (vx, 0, 0) \\ d_2(e_4) &= (0, 0, v) & d_2(ye_1) &= (vy, -vx, 0) \\ d_2(e_5) &= (0, 0, x) & d_2(ze_1) &= (vz, -vy, 0) \\ d_2(e_6) &= (0, 0, y) & d_2(ve_1) &= (0, vy, 0) \\ d_2(e_7) &= (0, 0, z) & d_2(ve_2) &= (\alpha^{-2}vx, vz, 0) \\ d_2(xe_4) &= (0, 0, vx) & d_2(xe_2) &= (\alpha^{-2}vy, -\alpha^{-1}vx, 0) \end{aligned}$$

where $e_1, \ldots, e_7$ denote the elements comprising the standard basis of R^7 as a free R-module. $\square$

If $f\colon M \to N$ is a homomorphism of R-modules, we let f^* represent the induced map $\operatorname{Hom}_R(f,R)\colon \operatorname{Hom}_R(N,R) \to \operatorname{Hom}_R(M,R)$. If (D,δ) is a complex of R-modules, then the complex (D^*,δ^*) has $(D^*)_i = (D_{-i})^*$ and differentials $(\delta^*)_i = (\delta_{-i})^*$. We write δ_i^* for $(\delta^*)_i$.

Note that, upon identification of R^* with R, the map $d_i^*\colon R^2 \to R^2$ for $i \geq 0$ is given in the standard basis of R^2 by the matrix

$$\begin{pmatrix} v & y \\ \alpha^i x & z \end{pmatrix}.$$

Similarly, the maps d_{-1}^* and d_{-2}^* are given by the transposes of the matrices defining d_1 and d_2, respectively.

1.5. **Lemma.** *The complex*

$$C^*: \quad \cdots \to R^2 \xrightarrow{d_2^*} R^2 \xrightarrow{d_1^*} R^2 \xrightarrow{d_0^*} R^2 \xrightarrow{d_{-1}^*} R^3 \xrightarrow{d_{-2}^*} R^7 \to \cdots$$

satisfies $H_i(C^*) = 0$ if and only if $i \geq 1$.

Proof. As a k-vector space, $\operatorname{Im} d_i^*$ for $i \geq 0$ is generated by the following elements:

$$\begin{aligned}
d_i^*(1,0) &= (v, \alpha^i x) & d_i^*(z,0) &= (vz, -\alpha^{i-1} vx)\\
d_i^*(0,1) &= (y,z) & d_i^*(0,v) &= (vy, vz)\\
d_i^*(v,0) &= (0, \alpha^i vx) & d_i^*(0,x) &= (0, -\alpha^{-1} vx)\\
d_i^*(x,0) &= (vx, \alpha^i vy) & d_i^*(0,y) &= (-vx, -vy)\\
d_i^*(y,0) &= (vy, 0) & d_i^*(0,z) &= (-vy, 0)
\end{aligned}$$

One can see therefore that $\operatorname{rank}_k(\operatorname{Im} d_i^*) = 8$ if $i \geq 1$ and $\operatorname{rank}_k(\operatorname{Im} d_0^*) = 7$. This implies that $\operatorname{rank}_k(\operatorname{Ker} d_i^*) = 8$ if $i \geq 1$ and $\operatorname{rank}_k(\operatorname{Ker} d_0^*) = 9$, and it follows that $H_i(C^*) = 0$ for all $i \geq 1$ and $H_0(C^*) \neq 0$.

For the proof that $H_i(C^*) \neq 0$ for $i = -1, -2$, note that the image of d_{-1}^* is generated as a k-vector space by the following elements.

$$\begin{aligned}
d_{-1}^*(1,0) &= (v, \alpha^{-1}x, yz) & d_{-1}^*(z,0) &= (vz, -\alpha^{-2} vx, 0)\\
d_{-1}^*(0,1) &= (y,z,0) & d_{-1}^*(0,v) &= (vy, vz, 0)\\
d_{-1}^*(v,0) &= (0, \alpha^{-1} vx, 0) & d_{-1}^*(0,x) &= (0, -\alpha^{-1} vx, 0)\\
d_{-1}^*(x,0) &= (vx, \alpha^{-1} vy, 0) & d_{-1}^*(0,y) &= (-vx, -vy, 0)\\
d_{-1}^*(y,0) &= (vy, 0, 0) & d_{-1}^*(0,z) &= (-vy, 0, 0)
\end{aligned}$$

One sees easily that $\operatorname{rank}_k(\operatorname{Im} d_{-1}^*) \leq 8$. Therefore $\operatorname{rank}_k(\operatorname{Ker} d_{-1}^*) \geq 8$, and so $H_{-1}(C^*) \neq 0$. Clearly $\operatorname{rank}_k(\operatorname{Ker} d_{-2}^*)$ consists of at least nine linearly independent elements, namely the nine quadric elements in R_2^3. This shows that $H_{-2}(C^*) \neq 0$.

Finally, we note from the matrix representing d_2 that $\operatorname{Coker} d_2 \cong N \oplus k$, for some finite R-module N. Therefore $H_i(C^*) \cong \operatorname{Ext}_R^{-i-2}(N \oplus k, R) \neq 0$ for all $i \leq -3$, since R is not Gorenstein. $\square$

1.6. For each integer $s \geq 1$ let M_s be the cokernel of the map $d_{-s}\colon R^2 \to R^2$. Using Lemma 1.4, note that

$$M_s = \operatorname{Coker}(d_{-s}) \cong \operatorname{Im}(d_{-s-1}) = \operatorname{Ker}(d_{-s-2})$$

and a truncation of the complex C gives the beginning a minimal free resolution of the R-module M_s:

$$\cdots \to R^7 \xrightarrow{d_2} R^3 \xrightarrow{d_1} R^2 \xrightarrow{d_0} R^2 \xrightarrow{d_{-1}} R^2 \to \cdots \to R^2 \xrightarrow{d_{-s+1}} R^2 \xrightarrow{d_{-s}} R^2 \to M_s \to 0$$

The proof of Lemma 1.4 shows that M_s has Hilbert series $H_{M_s}(t) = 2t + 6t^2$.

We are now ready to state our main theorem:

1.7. **Theorem.** *For the family of R-modules $\{M_s\}_{s\geq 1}$ defined above, we have:*

(1) *M_s satisfies (TR_i) if and only if $i < s$.*
(2) *$\operatorname{Tr}(M_s)$ satisfies (TR_i) if and only if $i > -s$.*

Note that this contains the Theorem stated in the introduction: indeed, one can take there the modules M_s to be as above and $L = \operatorname{Tr}(M_1)$.

Proof of Theorem 1.7. The second part of the theorem follows from the first part and the simple observation that a finite R-module N satisfies the condition (TR_i) if and only if $\operatorname{Tr}(N)$ satisfies (TR_{-i}).

To compute $\operatorname{Ext}^*_R(N, R)$ for an R-module N we take a minimal free resolution of N, we apply $(-)^*$ to it, and then compute homology of the resulting complex.

Applying $(-)^*$ to the minimal free resolution of M_s given in 1.6, and identifying R with R^*, one obtains the complex

$$R^2 \xrightarrow{d_s^*} R^2 \xrightarrow{d_{s-1}^*} R^2 \to \cdots \to R^2 \xrightarrow{d_1^*} R^2 \xrightarrow{d_0^*} R^2 \xrightarrow{d_{-1}^*} R^3 \xrightarrow{d_{-2}^*} R^7 \to \cdots$$

Lemma 1.5 shows that $\operatorname{Ext}^i_R(M_s, R) = 0$ for all $1 \leq i \leq s-1$, and $\operatorname{Ext}^i_R(M_s, R) \neq 0$ for $i \geq s$.

A minimal free resolution of $\operatorname{Tr}(M_s)$ is given by

$$\cdots \to R^2 \xrightarrow{d_{s+1}^*} R^2 \xrightarrow{d_s^*} R^2,$$

and applying $(-)^*$ we get

$$R^2 \xrightarrow{d_{-s}} R^2 \xrightarrow{d_{-s-1}} R^2 \to \cdots .$$

Lemma 1.4 shows that $\operatorname{Ext}^i_R(\operatorname{Tr}(M_s), R) = 0$ for all $i \geq 0$. This establishes (1), and hence the entire theorem. □

2. Dependence

Theorem 1.7 shows that the conditions (TR_i) are, to a large extent, independent. However, in the Artinian graded case, there is some overlap between these conditions.

Let R be a standard graded ring and M a finite graded R-module. As noted by Avramov and Martsinkovsky in [4], the module M is totally reflexive if and only if it satisfies (TR_i) for all i, except possibly for $i = 1$ or $i = -1$.

This result is based on a formula obtained by Avramov, Buchweitz and Sally in [3]. Buchweitz pointed out to us that the same formula also yields the following.

2.1. **Proposition.** *Assume that R is an Artinian standard graded ring, and M is a finitely generated graded R-module. Let A be a finite set of integers of the same parity. If M satisfies (TR_i) for all $i \in \mathbb{Z} \smallsetminus A$, then the module M is totally reflexive.*

Proof. We may assume $0 \notin A$. Suppose that M satisfies (TR_i) for all $i \in \mathbb{Z} \setminus A$. Since A is a finite set, we conclude that M satisfies (TR_i) for all i with $|i| \gg 0$. This allows us to use the main formula in [3] which asserts equalities of rational functions

$$\sum_{n\in\{0\}\cup A} (-1)^n H_{\mathrm{Ext}^n_R(M,R)}(t) = \frac{H_M(t^{-1}) \cdot H_R(t)}{H_R(t^{-1})} \tag{1}$$

$$\sum_{-n\in\{0\}\cup A} (-1)^n H_{\mathrm{Ext}^n_R(M^*,R)}(t) = \frac{H_{M^*}(t^{-1}) \cdot H_R(t)}{H_R(t^{-1})} \tag{2}$$

We set

$$P(t) = \sum_{n\in A} H_{\mathrm{Ext}^n_R(M,R)}(t) \quad \text{and} \quad Q(t) = \sum_{-n\in A} H_{\mathrm{Ext}^n_R(M^*,R)}(t).$$

Let $\sigma = 0$ if A contains only odd integers and $\sigma = 1$ if A contains only even integers. The formulas (1) and (2) give then

$$H_{M^*}(t) = \frac{H_M(t^{-1}) \cdot H_R(t)}{H_R(t^{-1})} + (-1)^\sigma P(t)$$

$$H_{M^{**}}(t) = \frac{H_{M^*}(t^{-1}) \cdot H_R(t)}{H_R(t^{-1})} + (-1)^\sigma Q(t)$$

Substituting the formula for $H_{M^*}(t^{-1})$ given by the first equation into the second equation, we obtain:

$$H_{M^{**}}(t) = H_M(t) + \frac{H_R(t)}{H_R(t^{-1})} \cdot (-1)^\sigma P(t^{-1}) + (-1)^\sigma Q(t)$$

and it follows that

$$H_R(t) \cdot P(t^{-1}) + H_R(t^{-1}) \cdot Q(t) = (-1)^\sigma H_R(t^{-1})\big(H_{M^{**}}(t) - H_M(t)\big).$$

Assume that A contains only odd integers. We have then $-2 \notin A$, hence (TR_i) is satisfied for $i = -2$. The map $M \to M^{**}$ is then surjective, implying a coefficientwise inequality $H_{M^{**}}(t) \leq H_M(t)$. Since $\sigma = 0$ in this case and $H_R(t^{-1})$ has positive coefficients, it follows that the Laurent polynomial on the right has nonpositive coefficients.

Both terms of the left-hand side sum are Laurent polynomials with nonnegative coefficients, and it follows that $P(t) = 0$ and $Q(t) = 0$, implying that $\mathrm{Ext}^n_R(M,R) = 0$ for all $n \in A$ and $\mathrm{Ext}^n_R(M^*,R) = 0$ for all n with $-n \in A$. In conclusion, (TR_i) is satisfied for all $i \neq -1$. Furthermore, we conclude from the formula above that $H_{M^{**}}(t) = H_M(t)$. Since the map $M \to M^{**}$ is surjective, it follows that it is an isomorphism, hence (TR_i) is satisfied for $i = -1$ as well.

Proceed similarly when A contains only even integers. □

Proposition 2.1 leads to the following question:

Question. Let R be a commutative (local) Artinian ring. If a finite R-module M satisfies (TR_i) for all but finitely many values of i, does it follow that M is totally reflexive?

3. Minimimal acyclic complexes of free modules

Let S be a commutative Noetherian local ring with maximal ideal $\mathfrak{n}$.

A complex F of free S-modules

$$\cdots \to F_{i+1} \xrightarrow{\phi_{i+1}} F_i \xrightarrow{\phi_i} F_{i-1} \to \cdots$$

is said to be *minimal* if $\phi_i(F_i) \subseteq \mathfrak{n}F_{i-1}$ for all i. The complex F is *acyclic* if $H_i(F) = 0$ for all $i \in \mathbb{Z}$. For any minimal acyclic complex of finite free S-modules F we can consider two sequences:

$$\boldsymbol{\beta}_F^+ := \{\operatorname{rank}_R F_i\}_{i\geqslant 0} \quad \text{and} \quad \boldsymbol{\beta}_F^- := \{\operatorname{rank}_R F_{-i}\}_{i\geqslant 0}$$

Assuming that $F_i \neq 0$ for all i, it is then natural to ask whether these two sequences have similar asymptotic behavior.

A sequence $\{\beta_i\}_{i\geq 0}$ is said to have *exponential growth* if there exist numbers $1 < A \leq B$ such that inequalities $A^i \leq \beta_i \leq B^i$ hold for all $i \gg 0$.

When the maximal ideal of S satisfies $\mathfrak{n}^3 = 0$, Lescot [8] proved that the Betti numbers of a finitely generated S-module N are either eventually stationary, or they have exponential growth; in the last case they are eventually strictly increasing. It is clear from the Poincaré series given in 1.2 that the Betti numbers of k over our ring R have exponential growth. Furthermore, with d_2 as defined there, since $\operatorname{Coker}(d_2)$ has a copy of k as a direct summand, its Betti numbers have exponential growth and are eventually strictly increasing.

In conclusion, the complex C of Lemma 1.4 has the following properties:

(a) $\boldsymbol{\beta}_C^+$ has exponential growth and is eventually strictly increasing.
(b) $\boldsymbol{\beta}_C^-$ is constant (nonzero).

Several questions arise:

Question. Does there exist a ring S as above and a minimal acyclic complex of free nonzero S-modules F such that $\boldsymbol{\beta}_F^-$ has exponential growth (or is eventually strictly increasing) and $\boldsymbol{\beta}_F^+$ is eventually constant?

Question. Do there exist examples of different asymptotic behavior for $\boldsymbol{\beta}_F^-$ and $\boldsymbol{\beta}_F^+$ if we also require $H(F^*) = 0$? Can such examples exist over a Gorenstein ring?

The last question is equivalent to asking whether the Betti numbers of M and M^* can have different asymptotic behavior when M is totally reflexive, and, in particular, when S is Gorenstein. Theorem 5.6 of [2] shows that the answer to this question is "no" when S is a complete intersection.

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