FREE RESOLUTIONS OVER SHORT LOCAL RINGS

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To the memory of our friend and colleague Anders Frankild.

ABSTRACT. The structure of minimal free resolutions of finite modules M over commutative local rings $(R, \mathfrak{m}, k)$ with $\mathfrak{m}^3 = 0$ and $\operatorname{rank}_k(\mathfrak{m}^2) < \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$ is studied. It is proved that over generic R every M has a Koszul syzygy module. Explicit families of Koszul modules are identified. When R is Gorenstein the non-Koszul modules are classified. Structure theorems are established for the graded k-algebra $\operatorname{Ext}_R(k, k)$ and its graded module $\operatorname{Ext}_R(M, k)$.

INTRODUCTION

This paper is concerned with the structure of minimal free resolutions of finite (that is, finitely generated) modules over a commutative noetherian local ring R whose maximal ideal $\mathfrak{m}$ satisfies $\mathfrak{m}^3 = 0$. Over the last 30 years this special class has emerged as a testing ground for properties of infinite free resolutions. For a finite module M such properties are often stated in terms of its *Betti numbers* $\beta_n^R(M) = \operatorname{rank}_k \operatorname{Ext}_R^n(M, k)$, where $k = R/\mathfrak{m}$, or in terms of its *Poincaré series*

$$P_M^R(t) = \sum_{i=0}^{\infty} \beta_n^R(M)\, t^n \in \mathbb{Z}[\![t]\!] .$$

Patterns that had been conjectured not to exist at one time have subsequently been discovered over rings with $\mathfrak{m}^3 = 0$: Not finitely generated algebras $\operatorname{Ext}_R(k, k)$ (Roos, 1979); transcendental Poincaré series $P_k^R(t)$ (Anick, 1980); modules with constant Betti numbers and aperiodic minimal free resolutions (Gasharov and Peeva, 1990); families of modules with rational Poincaré series that admit no common denominator (Roos, 2005); reflexive modules M with $\operatorname{Ext}_R^n(M, R) = 0 \neq \operatorname{Ext}_R^n(\operatorname{Hom}_R(M, R), R)$ for all $n \geq 1$ (Jorgensen and Şega, 2006).

On the other hand, important conjectures on infinite free resolutions that are still open in general have been verified over rings with $\mathfrak{m}^3 = 0$: Each sequence $(\beta_n^R(M))_{n \geqslant 0}$ is eventually non-decreasing, and grows either polynomially or exponentially (Lescot, 1985). When M has infinite projective dimension one has $\operatorname{Ext}_R^n(M, M \oplus R) \neq 0$ for infinitely many n (Huneke, Şega and Vraciu, 2004).

The work presented below is motivated by an 'unusually high' incidence in the appearance of modules M with 'Koszul-like' behavior, exemplified by an equality

$$(*) \qquad P_M^R(t) = \frac{p_M(t)}{1 - et + rt^2}$$

with $e = \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$, $r = \operatorname{rank}_k(\mathfrak{m}^2)$, and $p_M(t) \in \mathbb{Z}[t]$.

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We provide structural explanations for the phenomenon and natural conditions for its occurrence. To describe our results, note that the property $\partial(F) \subseteq \mathfrak{m}F$ of a minimal free resolution F of M allows one to form for each $j \geq 0$ a complex

$$\operatorname{lin}_j(F) = \quad 0 \to \frac{F_j}{\mathfrak{m}F_j} \to \cdots \to \frac{\mathfrak{m}^{j-n}F_n}{\mathfrak{m}^{j+1-n}F_n} \to \cdots \to \frac{\mathfrak{m}^j F_0}{\mathfrak{m}^{j+1}F_0} \to \frac{\mathfrak{m}^j M}{\mathfrak{m}^{j+1}M} \to 0$$

of k-vector spaces. Following [14], we say that M is *Koszul* if every complex $\operatorname{lin}_j(F)$ is acyclic. When R is a graded k-algebra generated in degree 1 and M is a graded R-module generated in a single degree, say d, this means that M has a d-linear free resolution. The results in this paper are new also in this more restrictive setup.

Our main theorem states that if there exists an element $x \in \mathfrak{m}$ satisfying $x^2 = 0$ and $\mathfrak{m}^2 = x\mathfrak{m}$, then every finite R-module has a syzygy module that is Koszul; see Section 1. The crucial step is to find a quadratic hypersurface ring mapping onto R by a Golod homomorphism. Results of Herzog and Iyengar [14] then apply and yield, in particular, formula $(*)$.

We call an element x as above a *Conca generator* of $\mathfrak{m}$ because Conca [5] shows that such an x exists in generic standard graded k-algebras with $r \leq e - 1$.

Assuming that $\mathfrak{m}$ has a Conca generator, in Section 2 we provide a finite presentation of the algebra $\mathcal{E} = \operatorname{Ext}_R(k, k)$ and prove that for every finite R-module M the $\mathcal{E}$-module $\operatorname{Ext}_R(M, k)$ has a resolution of length at most 2 by finite free graded $\mathcal{E}$-modules. This yields information on the degree of the polynomial $p_M(t)$ in $(*)$.

The last two sections are taken up by searches for Koszul modules.

In Section 3 we prove that modules annihilated by a Conca generator are Koszul.

In Section 4, we identify the rings with $\mathfrak{m}^3 = 0$ and $\operatorname{rank}_k \mathfrak{m}^2 \leq 1$ whose maximal ideal $\mathfrak{m}$ has a Conca generator. The Gorenstein rings R with $e \geq 2$ are among them; in particular, they satisfy formula $(*)$, which is known from Sjödin [20]. We prove that over such rings the indecomposable non-Koszul modules are precisely the negative syzygies of k, and that these modules are characterized by their Hilbert function. For $e = 2$ this follows from Kronecker's classification of pairs of commuting matrices. It is unexpected that the classification extends *verbatim* to Gorenstein rings with $\mathfrak{m}^3 = 0$ and $e \geq 3$, which have wild representation type.

1. Minimal resolutions

In this paper the expression *local ring* $(R, \mathfrak{m}, k)$ refers to a commutative Noetherian ring R with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Such a ring R is called *Koszul* if its residue field is a Koszul R-module, as defined in the introduction; when R is standard graded this coincides with the classical notion. An *inflation* $R \to R'$ is a homomorphism to a local ring $(R', \mathfrak{m}', k')$ that makes R' into a flat R-module and satisfies $\mathfrak{m}' = R'\mathfrak{m}$.

Here is the main result of this section:

1.1. **Theorem.** *Let $(R, \mathfrak{m}, k)$ be a local ring.*

If for some inflation $R \to (R', \mathfrak{m}', k')$ the ideal $\mathfrak{m}'$ has a Conca generator, then R is Koszul, each finite R-module M has a Koszul syzygy module, and one has

$$P^R_k(t) = \frac{1}{1 - et + rt^2}\,; \tag{1.1.1}$$

$$P^R_M(t) = \frac{p_M(t)}{1 - et + rt^2} \quad \textit{with} \quad p_M(t) \in \mathbb{Z}[t]\,, \tag{1.1.2}$$

where we have set $e = \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$ *and* $r = \operatorname{rank}_k(\mathfrak{m}^2/\mathfrak{m}^3)$.

A notion of Conca generator was introduced above; we slightly expand it:

1.2. We say that x is a *Conca generator* of an ideal J if x is in J and satisfies $x \neq 0 = x^2$ and $xJ = J^2$. One then has $J^3 = (xJ)J = x^2J = 0$, and also $x \notin J^2$: The contrary would imply $x \in J^2 = xJ \subseteq J^3 = 0$, a contradiction.

When the ring R in the theorem is standard graded a result of Conca, Rossi, and Valla, see [7, (2.7)], implies that it is Koszul; the proof relies on the theory of Koszul filtrations developed by these authors. A direct proof is given by Conca [5, Lem. 2]. Neither argument covers the local situation, nor gives information on homological properties of R-modules other than k.

We prove Theorem 1.1 via a result on the structure of R. It is stated in terms of Golod homomorphisms, for which we recall one of several possible definitions.

1.3. Let $(Q, \mathfrak{q}, k)$ be a local ring. Let D be a minimal free resolution of k over Q that has a structure of graded-commutative DG algebra with $D_0 = Q$; such a resolution always exists, see [11].

Let $\varkappa\colon Q \to R$ be a surjective homomorphism of rings, and set $A = R \otimes_Q D$. For $a \in A$ let $|a| = n$ indicate $a \in A_n$, and set $\overline{a} = (-1)^{|a|+1}a$. Let $\boldsymbol{h}$ denote a homogeneous basis of the graded k-vector space $\mathrm{H}_{\geqslant 1}(A)$.

The homomorphism $\varkappa$ is *Golod* if there is a function $\mu\colon \bigsqcup_{m=1}^{\infty} \boldsymbol{h}^m \to A$ satisfying

$$\mu(h) \text{ is a cycle in the homology class of } h \text{ for each } h \in \boldsymbol{h}\,; \tag{1.3.1}$$

$$\partial\mu(h_1, \dots, h_m) = \sum_{i=1}^{m-1} \overline{\mu(h_1, \dots, h_i)}\mu(h_{i+1}, \dots, h_m) \quad \text{for each } m \geq 2\,; \tag{1.3.2}$$

$$\mu(\boldsymbol{h}^m) \subseteq \mathfrak{m}A \quad \text{for each } m \geq 1\,. \tag{1.3.3}$$

The following structure theorem is used multiple times in the paper.

1.4. **Theorem.** *Let* $(R, \mathfrak{m}, k)$ *be a local ring and* x *a Conca generator of* $\mathfrak{m}$.

There exist a regular local ring $(P, \mathfrak{p}, k)$, *a minimal set of generators* $u_1, \dots, u_e$ *of* $\mathfrak{p}$, *and a surjective homomorphism* $\pi\colon P \to R$, *with* $\operatorname{Ker}(\pi) \subseteq \mathfrak{p}^2$ *and* $\pi(u_e) = x$.

Any such π *induces a Golod homomorphism* $Q \to R$, *where* $Q = P/(u_e^2)$.

Indeed, R is artinian by 1.2, so Cohen's Structure Theorem yields a surjection $\pi\colon P \to R$ with $(P, \mathfrak{p}, k)$ regular local and $\operatorname{Ker}(\pi) \subseteq \mathfrak{p}^2$; now 1.2 gives $x \notin \mathfrak{m}^2$, hence any $u_e \in \mathfrak{p}$ with $\pi(u_e) = x$ extends to a minimal generating set of $\mathfrak{p}$. It remains to prove that $Q \to R$ is Golod. For this we use a construction going back to H. Cartan.

1.5. *Construction.* Form a complex of free Q-modules of rank one

$$D = \quad \cdots \longrightarrow Qy_n \xrightarrow{u_e} Qy_{n-1} \longrightarrow \cdots \longrightarrow Qy_1 \xrightarrow{u_e} Qy_0 \longrightarrow 0 \longrightarrow \cdots$$

where y_n is a basis element in degree n. The multiplication table

$$y_{2i}y_{2j} = \binom{i+j}{i} y_{2(i+j)} = y_{2j}y_{2i}$$

$$y_{2i}y_{2j+1} = \binom{i+j}{i} y_{2(i+j)+1} = y_{2j+1}y_{2i}$$

$$y_{2i+1}y_{2j+1} = 0 = y_{2j+1}y_{2i+1}$$

for all $i, j \geq 0$ turns D into a graded-commutative DG Q-algebra with unit y_0.

1.6. **Lemma.** *Let C be the Koszul complex on the image in Q of $\{u_1,\dots,u_{e-1}\}$.*
The DG Q-algebra $C\otimes_Q D$ is a minimal free resolution of k.
The DG R-algebra $A=B\otimes_Q D$, where $B=R\otimes_Q C$, satisfies the conditions:

$$\partial(A_1)=\mathfrak{m}\subseteq R=A_0\,,\quad \mathfrak{m}^2A_{\geqslant 1}\subseteq\partial(\mathfrak{m}A_{\geqslant 2})\,,\quad \textit{and}\quad \mathrm{Z}_{\geqslant 1}(A)\subseteq \mathfrak{m}A_{\geqslant 1}\,.$$

Proof. The canonical augmentation $D\to Q/u_eQ$ is a quasi-isomorphism. It induces the first quasi-isomorphism below, because C is a bounded complex of free Q-modules:

$$C\otimes_Q D\simeq C\otimes_Q (Q/u_eQ)\simeq k\,.$$

The second one holds because the sequence $u_1,\dots,u_{e-1}$ is (Q/u_eQ)-regular.

The relations $\partial(A_1)=\mathfrak{m}\subseteq R=A_0$ hold by construction.

Every element $a\in A_n$ can be written uniquely in the form $a=\sum_{i=0}^n b_i\otimes y_i$ with $b_i\in B_{n-i}$. Supposing that it is a cycle, one gets

$$\partial(b_i)=(-1)^{n-i}xb_{i+1}\quad\text{for}\quad i=0,\dots,n-1\quad\text{and}\quad \partial(b_n)=0\,.$$

If $b\in B$ satisfies $\partial(b)\in\mathfrak{m}^2B$, then b is in $\mathfrak{m}B$; see [17, Cor 2.1]. Thus, $\partial(b_n)=0$ implies $b_n\in\mathfrak{m}B$. Assuming by descending induction that $b_j\in\mathfrak{m}B$ for some $j\leq n$, the equalities above imply $\partial(b_{j-1})\in\mathfrak{m}^2B$, whence $b_{j-1}\in\mathfrak{m}B$, and finally $a\in\mathfrak{m}A$.

It remains to verify the condition $\mathfrak{m}^2A\subseteq\partial(\mathfrak{m}A)$. One has $B_{\leqslant e-1}=B$, so to obtain the desired inclusion we induce on n to show that the following holds:

$$\mathfrak{m}^2(B_n\otimes_Q D)\subseteq\partial(\mathfrak{m}(B_{\leqslant n}\otimes_Q D))\quad\text{for each } n\,.$$

For $n\leq -1$ the formula above holds because $B_{<0}=0$, so assume it holds for some integer $n\geq -1$. As B and D are complexes of free modules, one gets relations:

$$\begin{aligned}
x(B_{n+1}\otimes_Q D)&=B_{n+1}\otimes_Q u_eD\\
&\subseteq B_{n+1}\otimes_Q \partial(D)\\
&\subseteq \partial(B_{n+1})\otimes_Q D+\partial(B_{n+1}\otimes_Q D)\\
&\subseteq \mathfrak{m}(B_n\otimes_Q D)+\partial(B_{n+1}\otimes_Q D)\,.
\end{aligned}$$

The resulting inclusion provides the second link in the following chain:

$$\begin{aligned}
\mathfrak{m}^2(B_{n+1}\otimes_Q D)&=\mathfrak{m}x(B_{n+1}\otimes_Q D)\\
&\subseteq \mathfrak{m}^2(B_n\otimes_Q D)+\mathfrak{m}\partial(B_{n+1}\otimes_Q D)\\
&\subseteq \partial(\mathfrak{m}(B_{\leqslant n}\otimes_Q D))+\partial(\mathfrak{m}(B_{n+1}\otimes_Q D))\\
&=\partial(\mathfrak{m}(B_{\leqslant n+1}\otimes_Q D))\,.
\end{aligned}$$

The induction hypothesis gives the third link, so the argument is complete. □

Proof of Theorem 1.4. We construct a function μ as in 1.3 by induction on m.

To start, choose for each $h\in\boldsymbol{h}$ a cycle z_h in the homology class of h; the last inclusion in Lemma 1.6 yields $z_h\in\mathfrak{m}A$. Suppose that μ has been defined on $\boldsymbol{h}^{m-1}$ for some integer $m\geq 2$. The induction hypothesis then provides an element

$$\sum_{i=1}^{m-1}\overline{\mu(h_1,\dots,h_i)}\mu(h_{i+1},\dots,h_m)\in\mathfrak{m}^2A\,.$$

As $\mathfrak{m}^2A\subseteq\partial(\mathfrak{m}A)$ holds, by Lemma 1.6, the element above is the boundary of some element in $\mathfrak{m}A$, which we name $\mu(h_1,\dots,h_m)$. The induction step is complete. □

We will use an adaptation of a construction of Golod proposed by Gulliksen:

1.7. *Construction.* For each integer $n \geq 1$ set $\boldsymbol{h}_n = \{h \in \boldsymbol{h} : |h| = n\}$, and choose a free R-module V_n with basis $\{v_h\}_{h \in \boldsymbol{h}_n}$. The free R-modules

$$G_n = \bigoplus_{l+m+n_1+\cdots+n_m=n} A_l \otimes_R V_{n_1} \otimes_R \cdots \otimes_R V_{n_m} \tag{1.7.1}$$

and the R-linear maps $\partial\colon G_n \to G_{n-1}$ defined by the formula

$$\begin{aligned} \partial(a \otimes v_{h_1} \otimes \cdots \otimes v_{h_m}) &= \partial(a) \otimes v_{h_1} \otimes \cdots \otimes v_{h_m} \\ &\quad + (-1)^{|a|} \sum_{i=1}^{m} a\mu(h_1, \dots, h_i) \otimes v_{h_{i+1}} \otimes \cdots \otimes v_{h_m} \end{aligned} \tag{1.7.2}$$

then form a minimal free resolution of k over R; see [12, Prop. 1].

For a finite R-module M and $n \geq 0$ set $\Omega_n^R(M) = \operatorname{Coker}(\partial_{n+1}^F)$, where F is a minimal free resolution of M; up to isomorphism, $\Omega_n^R(M)$ does not depend on F.

1.8. *Remark.* Let $R \to (R', \mathfrak{m}', k')$ be an inflation.

Let M be a finite R-module and set $M' = M \otimes_R R'$.

For each $n \geq 0$ one has $\mathfrak{m}'^n M'/\mathfrak{m}'^{n+1} M' \cong k' \otimes_k (\mathfrak{m}^n M/\mathfrak{m}^{n+1} M)$.

If F is a minimal free resolution of M over R, then $R' \otimes_R F$ is one of M' over R', so $P_M^R(t) = P_{M'}^{R'}(t)$ and $\Omega_n^{R'}(M') \cong R' \otimes_R \Omega_n^R(M)$. The complexes $\operatorname{lin}_j(F)$ from the introduction satisfy $\operatorname{lin}_j^{R'}(R' \otimes_R F) \cong k' \otimes_k \operatorname{lin}_j^R(F)$ for each j. Thus, $\operatorname{lin}_j(R' \otimes_R F)$ and $\operatorname{lin}_j(F)$ are exact simultaneously, so M is Koszul if and only if M' is.

1.9. *Remark.* Set $H_M(t) = \sum_{n=0}^{\infty} \operatorname{rank}_k(\mathfrak{m}^n M/\mathfrak{m}^{n+1} M) \cdot t^n$.

When R is a Koszul ring with $\mathfrak{m}^l = 0$ for some integer l, and $N = \Omega_i^R(M)$ is a Koszul module the proof of [14, (1.8)] shows that $P_M^R(t) \cdot H_R(-t) - t^i H_N(-t)$ is a polynomial of degree at most $i + l - 2$.

Proof of Theorem 1.1. In view of Remark 1.8, we may replace R with R' and M with M', and so assume that the ideal $\mathfrak{m}$ itself has a Conca generator.

We start by proving that the ring R is Koszul. By [21, (7.5)], it suffices to show that for all integers $n \geq 1$ and $j \geq 1$ the map

$$\operatorname{Tor}_n^R(\iota_j, k)\colon \operatorname{Tor}_n^R(\mathfrak{m}^{j+1}, k) \to \operatorname{Tor}_n^R(\mathfrak{m}^j, k),$$

induced by the inclusion $\iota_j\colon \mathfrak{m}^{j+1} \to \mathfrak{m}^j$ is equal to 0. Because we have $\mathfrak{m}^3 = 0$, see 1.2, this boils down to proving $\operatorname{Tor}_n^R(\iota_1, k) = 0$ for all $n \geq 1$. These maps are the homomorphisms induced in homology by the inclusion $\mathfrak{m}^2 G \subseteq \mathfrak{m}G$, where G is some free resolution of k over R. We choose the one from Construction 1.7.

As seen from (1.7.1), the graded R-module $\mathfrak{m}^2 G$ is spanned by elements

$$g = ra \otimes v_{h_1} \otimes \cdots \otimes v_{h_m}$$

with $r \in \mathfrak{m}^2$, $a \in A$, and $h_i \in \boldsymbol{h}$. By induction on m we prove $g \in \partial(\mathfrak{m}G)$. Lemma 1.6 settles the case $m = 0$. Suppose $m \geq 1$. The base case yields $ra = \partial(b)$ for some $b \in \mathfrak{m}A$, so from (1.7.2) one obtains:

$$g = \partial(b) \otimes v_{h_1} \otimes \cdots \otimes v_{h_m} = \partial(b \otimes v_{h_1} \otimes \cdots \otimes v_{h_m}) - (-1)^{|b|} c,$$

$$\text{where} \quad c = \sum_{i=1}^{m} b\mu(h_1, \dots, h_i) \otimes v_{h_{i+1}} \otimes \cdots \otimes v_{h_m}.$$

One has $\mu(h_1, \dots, h_i) \in \mathfrak{m}A$, see (1.3.3), so c is in $\mathfrak{m}^2 G$. The induction hypothesis now yields $c \in \partial(\mathfrak{m}G)$, so one gets $g \in \partial(\mathfrak{m}G)$, and the inductive proof is complete.

Theorem 1.4 yields a hypersurface ring Q and a Golod homomorphism $Q \to R$, so M has a Koszul syzygy by [14, (5.9)]. Formulas (1.1.2) and (1.1.1) now follow from Remark 1.9 because one has $\mathfrak{m}^n = 0$ for $n \geq 3$; see 1.2. □

1.10. *Remark.* Let k be an algebraically closed field. The quotient algebras of the polynomial ring $k[X_1, \dots, X_e]$ by $\binom{e+1}{2} - r$ linearly independent quadratic forms are parametrized by the points of an appropriate Grassmannian. Conca proves that it contains a non-empty Zariski open subset, whose points correspond to algebras R for which the ideal $(X_1, \dots, X_e)R$ has a Conca generator; see [5, Thm. 10].

A standard graded ring $R = k[X_1, \dots, X_e]/I$ is said to be *G-quadratic* if the ideal I is generated by a Gröbner basis of quadrics with respect to some system of coordinates and some term order. The following implications hold, see [5, Lem. 2] for a proof of the first one and [4, 2.2] for one of the second:

$$(X_1, \dots, X_e)R \text{ has a Conca generator} \implies R \text{ is G-quadratic} \implies R \text{ is Koszul}\,.$$

To show that the conclusions of Theorem 1.1 may fail over G-quadratic rings, even when $r \leq e - 1$ holds, we use a general change-of-rings formula:

1.11. *Remark.* When R is the fiber product $S \times_k T$ of local rings $(S, \mathfrak{s}, k)$ and $(T, \mathfrak{t}, k)$ Dress and Krämer [8, Thm. 1] show that each finite S-module M satisfies

$$P^R_M(t) = \frac{P^S_M(t) \cdot P^T_k(t)}{P^S_k(t) + P^T_k(t) - P^S_k(t) P^T_k(t)}\,.$$

1.12. **Example.** For a field k, an integer $e \geq 5$, and indeterminates $X_1, \dots, X_e$ set

$$R = \frac{k[X_1, \dots, X_e]}{(X_1, X_2)^2 + (X_3, X_4)^2 + (X_5, \dots, X_e)(X_1, \dots, X_e)}\,.$$

This k-algebra is evidently G-quadratic, local and has $H_R(t) = 1 + et + 4t^2$. It is isomorphic to the fiber product $S \times_k T$ of the Koszul algebras

$$S = \frac{k[X_1, \dots, X_4]}{(X_1, X_2)^2 + (X_3, X_4)^2} \quad \text{and} \quad T = \frac{k[X_5, \dots, X_e]}{(X_5, \dots, X_e)^2}\,.$$

Thus, $P^S_k(t) = (1 - 4t + 4t^2)^{-1}$ and $P^T_k(t) = (1 - (e-4)t)^{-1}$, so Remark 1.11 yields

$$P^R_M(t) = P^S_M(t) \cdot \frac{1 - 4t + 4t^2}{1 - et + 4t^2}\,.$$

Roos [18, §5] produces an infinite family of S-modules whose Poincaré series admit no common denominator, so the conclusions of Theorem 1.1 fail for R.

2. Structure of cohomology

In this section we obtain precise and explicit information on the graded k-algebra $\operatorname{Ext}_R(k,k)$ and its graded left module $\operatorname{Ext}_R(M,k)$, assuming that $\mathfrak{m}$ has a Conca generator. We start by recalling terminology and introducing notation.

Let G be a free resolution of k over R. We identify $\operatorname{Ext}_R(k,k)$ with the cohomology, $\mathcal{E}$, of the complex $\operatorname{Hom}_R(G,G)$. Composition gives $\operatorname{Hom}_R(G,G)$ a structure of a DG algebra, so $\mathcal{E}$ inherits a structure of graded algebra over $\mathcal{E}^0 = k$.

2.1. Let $(R, \mathfrak{m}, k)$ be a local ring and x a Conca generator of $\mathfrak{m}$.

Multiplication with x induces a surjection $\upsilon \colon \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}^2$ with $\upsilon(x) = 0$. Thus, one can choose $x_1, \dots, x_e \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ so that $x_1 x, \dots, x_r x$ form a k-basis of $\mathfrak{m}^2$, the images of $x_{r+1},\dots,x_e$ in $\mathfrak{m}/\mathfrak{m}^2$ form one of $\operatorname{Ker}(\upsilon)$, and $x_e = x$. It follows that $x_1,\dots,x_e$ minimally generate $\mathfrak{m}$ and uniquely define elements $a_h^{ij} \in k$ satisfying

$$x_i x_j = \sum_{h=1}^{r} a_h^{ij} x_h x_e \quad \text{for} \quad 1 \le i \le j \le e-1\,. \tag{2.1.1}$$

Let $\xi_1,\dots,\xi_e$ denote the basis of $\mathcal{E}^1$ dual to the basis $x_1,\dots,x_e$ of $\mathfrak{m}/\mathfrak{m}^2$, under the canonical isomorphism $\mathcal{E}^1 \cong \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$.

We let $k\{\xi_1,\dots,\xi_e\}$ denote the non-commutative polynomial ring on indeterminates $\xi_1,\dots,\xi_e$, set $[\xi_i,\xi_j] = \xi_i\xi_j + \xi_j\xi_i$, and write $(\phi_1,\dots,\phi_r)$ for the two-sided ideal generated by non-commutative polynomials $\phi_1,\dots,\phi_r$. A graded module $\mathcal{M}$ is said to be *bounded below* if $\mathcal{M}^n = 0$ holds for all $n \ll 0$.

In the proof of the next theorem, and later on, we use non-traditional notation for associated graded objects: For a local ring $(R,\mathfrak{m},k)$ and an R-module M, set

$$R^{\mathrm{g}} = \bigoplus_{j\geqslant 0} \mathfrak{m}^j/\mathfrak{m}^{j+1} \quad \text{and} \quad M^{\mathrm{g}} = \bigoplus_{j\geqslant 0} \mathfrak{m}^j M/\mathfrak{m}^{j+1}M\,.$$

Of course, R^{g} is a graded ring and M^{g} is a graded module over it.

2.2. **Theorem.** *With the hypotheses and notation above the following hold.*

(1) *There is an isomorphism of graded k-algebras*

$$\mathcal{E} \cong \frac{k\{\xi_1,\dots,\xi_e\}}{(\phi_1,\dots,\phi_r)}\,, \quad \textit{where}$$

$$\phi_h = [\xi_h,\xi_e] + \sum_{1\le i<j\le e-1} a_h^{ij}[\xi_i,\xi_j] + \sum_{i=1}^{e-1} a_h^{ii}\xi_i^2 \quad \textit{for} \quad h = 1,\dots,r\,,$$

and a_h^{ij} are the elements in k defined by formulas (2.1.1).

(2) *The images of the words in $\{\xi_1,\dots,\xi_e\}$ that contain no subword from the set $\{\xi_e\xi_1,\dots,\xi_e\xi_r\}$ form a basis of $\mathcal{E}$ over k.*

(3) *Each bounded below (respectively, finitely presented) graded $\mathcal{E}$-module has a resolution of length* 2 *by bounded below (respectively, finite) graded free modules.*

Proof of Theorem 2.2. (1) and (2). As R is Koszul, the k-algebra $\mathcal{E}$ is generated by $\mathcal{E}^1$, see [21, (7.5)], so there is a surjective map $k\{\xi_1,\dots,\xi_e\} \to \mathcal{E}$ of graded k-algebras. Sjödin, [19, Thm. 4] shows that each ϕ_h is in its kernel, so it induces a surjective homomorphism $\mathcal{D} \to \mathcal{E}$, where we have set $\mathcal{D} = k\{\xi_1,\dots,\xi_e\}/(\phi_1,\dots,\phi_r)$.

Order the words in $\xi_1,\dots,\xi_e$ first by degree, and within each degree lexicographically with respect to $\xi_e > \cdots > \xi_1$. This order is admissible in the sense of [22]. By [22, Thm. 7], a spanning set of $\mathcal{D}$ over k is given by those words that do not contain as a subword the leading monomial of any element from the ideal $(\phi_1,\dots,\phi_r)$. The leading term of ϕ_i is $\xi_e\xi_i$ so $\mathcal{D}$ is spanned, *a fortiori*, by the words containing no subword $\xi_e\xi_i$ for $i = 1,\dots,r$. Call such a word reduced. Let w_n denote the number of reduced words of degree n; for $1 \le i \le e$, let $w_{n,i}$ denote the number of reduced words of degree n ending in ξ_i. One then has

$$\begin{aligned} w_{n,i} &= w_{n-1} && \text{for } r+1 \le i \le e \text{ and } n \ge 1\,;\\ w_{n,i} &= w_{n-1} - w_{n-1,e} && \text{for } \quad 1 \le i \le r \text{ and } n \ge 2\,. \end{aligned}$$

For every $n \geq 2$ the definitions and the relations above yield

$$\begin{aligned} w_n &= \sum_{i=1}^{r} w_{n,i} + \sum_{i=r+1}^{e} w_{n,i} \\ &= r(w_{n-1} - w_{n-2}) + (e-r)w_{n-1} \\ &= ew_{n-1} - rw_{n-2}\,. \end{aligned}$$

These equalities, along with $w_0 = 1$ and $w_1 = e$, imply the second equality below:

$$\begin{aligned} \sum_{n=0}^{\infty} \operatorname{rank}_k(\mathcal{E}^n)t^n &= \frac{1}{1-et+rt^2} \\ &= \sum_{n=0}^{\infty} w_n t^n \\ &\succcurlyeq \sum_{n=0}^{\infty} \operatorname{rank}_k(\mathcal{D}^n)t^n \\ &\succcurlyeq \sum_{n=0}^{\infty} \operatorname{rank}_k(\mathcal{E}^n)s^n\,. \end{aligned}$$

The first equality is (1.1.1), and the coefficient-wise inequalities $\succcurlyeq$ of formal power series are evident. Thus, all the relations above are equalities. It follows that the homomorphism $\mathcal{D} \to \mathcal{E}$ is bijective and the reduced words form a basis.

(3) The graded k-algebra R^{g} is Koszul, by the definition of the Koszul property for R. Thus, the k-algebra $\operatorname{Ext}_{R^{\mathsf{g}}}(k,k)$ is generated by $\operatorname{Ext}^1_{R^{\mathsf{g}}}(k,k)$. As $\mathfrak{m}^3 = 0$ holds, Löfwall [17, Thm. 2.3] gives $\operatorname{Ext}_{R^{\mathsf{g}}}(k,k) \cong \mathcal{E}$ as graded k-algebras. Koszul duality yields $\operatorname{Ext}_{\mathcal{E}}(k,k) \cong R^{\mathsf{g}}$, see, for example, [14, (4.1)], and hence $\operatorname{Ext}^n_{\mathcal{E}}(k,k) \cong R^{\mathsf{g}}_n = 0$ for $n > 2$. This implies that each bounded below graded $\mathcal{E}$-module has a resolution of length 2 by bounded below graded free modules; see [3, §8, Prop. 8, Cor. 5 and Cor. 2].

Theorem 1.4 yields a Golod homomorphism from a hypersurface ring onto R. By a result of Backelin and Roos, see [2, Thm. 4, Cor. 3], this implies that $\mathcal{E}$ is coherent. It follows that when a graded $\mathcal{E}$-module $\mathcal{M}$ is finitely presented the kernel of the map in any finite free presentation is a finite free $\mathcal{E}$-module. □

Let M be an R-module, F a free resolution of M, and G one of k. Composition turns $\operatorname{Hom}_R(F,G)$ into a DG module over the DG algebra $\operatorname{Hom}_R(G,G)$. Thus, $\operatorname{Ext}_R(M,k) = \operatorname{H}(\operatorname{Hom}_R(F,G))$ is a graded left module over $\mathcal{E} = \operatorname{H}(\operatorname{Hom}_R(G,G))$ with $\operatorname{Ext}^0_R(M,k) = \operatorname{Hom}_R(M,k)$, and hence one has a natural homomorphism

$$\mathcal{E} \otimes_k \operatorname{Hom}_R(M,k) \longrightarrow \operatorname{Ext}_R(M,k)$$

of left $\mathcal{E}$-modules, which is an isomorphism when $\mathfrak{m}M = 0$.

We index graded objects following custom and convenience: For the nth component of a graded k-vector space V we write either V_n or V^{-n}. Let $\Sigma^i V$ denote the graded vector space with $(\Sigma^i V)_n = V_{n-i}$ for $n \in \mathbb{Z}$; equivalently, $(\Sigma^i V)^n = V^{n+i}$.

2.3. *Construction.* The R^{g}-module structure on M^{g} defines a k-linear map

$$\nu\colon R^{\mathsf{g}}_1 \otimes_k M^{\mathsf{g}}_0 \longrightarrow M^{\mathsf{g}}_1 \quad \text{with} \quad \overline{a} \otimes \overline{x} \longmapsto \overline{ax}\,.$$

Set $-^* = \operatorname{Hom}_R(-,k)$. As one has $\mathcal{E}^1 = (R_1^{\mathsf{g}})^*$, one gets a k-linear map

$$(M_1^{\mathsf{g}})^* \xrightarrow{\nu^*} (R_1^{\mathsf{g}} \otimes_k M_0^{\mathsf{g}})^* \cong (R_1^{\mathsf{g}})^* \otimes_k (M_0^{\mathsf{g}})^* = \mathcal{E}^1 \otimes_k (M_0^{\mathsf{g}})^* .$$

As $\xi_1, \dots, \xi_e$ is a k-basis for $\mathcal{E}^1$, for each $\psi \in (M_1^{\mathsf{g}})^*$ there are uniquely defined elements $\psi_1, \dots, \psi_e$ in $(M_0^{\mathsf{g}})^*$ such that $\nu^*(\psi) = \sum_{i=1}^{e} \xi_i \otimes \psi_i$. Evidently, the map

$$\delta \colon \mathcal{E} \otimes_k \Sigma^{-1}(M_1^{\mathsf{g}})^* \longrightarrow \mathcal{E} \otimes_k (M_0^{\mathsf{g}})^* \quad \text{with} \quad \delta(\xi \otimes \psi) = \sum_{i=1}^{e} \xi\xi_i \otimes \psi_i$$

is an $\mathcal{E}$-linear homomorphism of degree zero.

2.4. **Theorem.** *Let $(R, \mathfrak{m}, k)$ be a local ring such that $\mathfrak{m}$ has a Conca generator, let M be a finite R-module with $\mathfrak{m}^2 M = 0$, and set*

$$\mathcal{F} = \operatorname{Ker}(\delta) .$$

The graded $\mathcal{E}$-module $\mathcal{F}$ then is finite free, it satisfies

(2.4.1) $$\mathcal{F}^n = 0 \quad \text{for} \quad n \leq 0 ;$$

(2.4.2) $$\min\{j \geq 0 \mid (k \otimes_{\mathcal{E}} \mathcal{F})^{\geqslant j+2} = 0\} = \min\{n \geq 0 \mid \Omega_n^R(M) \text{ is Koszul}\} ,$$

and the following sequence is a minimal free resolution of $\operatorname{Ext}_R(M,k)$ over $\mathcal{E}$:

(2.4.3) $$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \otimes_k \Sigma^{-1}(M_1^{\mathsf{g}})^* \xrightarrow{\left[\begin{smallmatrix}\delta\\0\end{smallmatrix}\right]} \mathcal{E} \otimes_k (M_0^{\mathsf{g}})^* \oplus \Sigma^1\mathcal{F} \longrightarrow 0$$

2.5. *Remark.* The proof shows that the conclusions above, except for the finiteness of $\mathcal{F}$ over $\mathcal{E}$, hold when the hypothesis that R has a Conca generator is weakened to assuming that R is Koszul and $\mathfrak{m}^3 = 0$; see also Roos [18, (3.1)] for (2.4.3).

Proof of Theorem 2.4. Since $\mathcal{E}^n = 0$ for $n < 0$, it follows from the definition of δ that $\mathcal{F}^n = 0$ for $n < 0$; see 2.3. Moreover, δ^0 is the map $\nu^* \colon M_1^{\mathsf{g}*} \to R_1^{\mathsf{g}*} \otimes_k M_0^{\mathsf{g}*}$, which is injective, since ν is surjective. Thus $\mathcal{F}^0 = 0$ as well. Thus (2.4.1) is proved.

Set $\mathcal{M} = \operatorname{Ext}_R(M,k)$. The exact sequence of R-modules

$$0 \longrightarrow M_1^{\mathsf{g}} \longrightarrow M \longrightarrow M_0^{\mathsf{g}} \longrightarrow 0$$

induces an exact sequence of graded left $\mathcal{E}$-modules

$$\Sigma^{-1} \operatorname{Ext}_R(M_1^{\mathsf{g}}, k) \xrightarrow{\eth} \operatorname{Ext}_R(M_0^{\mathsf{g}}, k) \longrightarrow \mathcal{M} \longrightarrow$$

$$\operatorname{Ext}_R(M_1^{\mathsf{g}}, k) \xrightarrow{\Sigma^1\eth} \Sigma^1 \operatorname{Ext}_R(M_0^{\mathsf{g}}, k)$$

For $j = 0,1$ one has $\operatorname{Ext}_R(M_j^{\mathsf{g}}, k) \cong \mathcal{E} \otimes_k (M_j^{\mathsf{g}})^*$ as graded left $\mathcal{E}$-modules.

Using Proposition 2.7 below, it is not hard to check that one can replace the map $\eth$ with δ. Theorem 2.2(3) then implies that $\mathcal{F}$ is finite projective over $\mathcal{E}$. Bounded below projective graded $\mathcal{E}$-modules are free, see [3, §8, Prop. 8, Cor. 1], so the exact sequence above yields the free resolution (2.4.3). From this resolution one gets

$$\max\{j \geq 0 \mid \operatorname{Tor}_n^{\mathcal{E}}(k, \mathcal{M})_{n+j} \neq 0 \text{ for some } n\} = \min\{j \geq 0 \mid (k \otimes_{\mathcal{E}} \mathcal{F})^{\geqslant j+2} = 0\} .$$

By [14, (5.4)], the number on the left-hand side of the equality above is equal to the least integer $n \geq 0$, for which the R-module $\Omega_n^R(M)$ is Koszul, so we are done. □

2.6. **Corollary.** *For every finite R-module L the graded $\mathcal{E}$-module $\operatorname{Ext}_R(L,k)$ has a finite free resolution of length at most* 2.

Proof. Let $R^m \to L$ be a free cover. The exact sequence of R-modules

$$0 \longrightarrow K \longrightarrow R^m \longrightarrow L \longrightarrow 0$$

induces an exact sequence of graded $\mathcal{E}$-modules

$$0 \longrightarrow \Sigma^{-1}\operatorname{Ext}_R(K,k) \longrightarrow \operatorname{Ext}_R(L,k) \longrightarrow k^m \longrightarrow 0$$

One has $\mathfrak{m}^2 K \subseteq \mathfrak{m}^2(\mathfrak{m}R^m) = \mathfrak{m}^3 R^m = 0$, so the graded $\mathcal{E}$-module $\operatorname{Ext}_R(K,k)$ is finitely presented by the theorem. The ideal $\mathcal{E}^{\geqslant 1}$ of $\mathcal{E}$ is finitely generated, by Theorem 2.2(3), so the $\mathcal{E}$-module k is finitely presented, and hence so is k^m. It follows that $\operatorname{Ext}_R(L,k)$ is finitely presented. Now refer to Theorem 2.2(3). □

We have deferred to the end of the section the statement and proof of a simple general homological fact, for which we could not find an adequate reference.

2.7. **Proposition.** *Let R be a commutative ring, M an R-module, I an ideal, and $\nu\colon R_1 \otimes_{R_0} M_0 \to M_1$ the natural map, where $R_j = I^j/I^{j+1}$ and $M_j = I^jM/I^{j+1}M$.*

If M_0 is free over R_0, then there is a commutative diagram of R_0-linear maps

$$\begin{array}{ccc}
\operatorname{Hom}_R(M_1,R_0) & \xrightarrow{\quad\eth\quad} & \operatorname{Ext}^1_R(M_0,R_0) \\
{\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\cong} \\
\operatorname{Hom}_{R_0}(M_1,R_0) & \xrightarrow{\operatorname{Hom}_{R_0}(\nu,R_0)} & \operatorname{Hom}_{R_0}(R_1 \otimes_{R_0} M_0,R_0)
\end{array}$$

where $\eth$ is the connecting homomorphism defined by the exact sequence

$$0 \longrightarrow M_1 \longrightarrow M/I^2M \longrightarrow M_0 \longrightarrow 0$$

Proof. The hypothesis allows us to choose a surjective homomorphism of R-modules $F \to M_0$ with F free, so that the induced map $F/IF \to M_0$ is an isomorphism.

The exact sequence above appears in a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \longrightarrow & M/I^2M & \longrightarrow & M_0 & \longrightarrow & 0 \\
 & & {\scriptstyle\nu}\big\uparrow & & & & \| & & \\
 & & R_1 \otimes_{R_0} M_0 & & \big\uparrow & & \| & & \\
 & & \big\uparrow & & & & \| & & \\
0 & \longrightarrow & I \otimes_R F & \longrightarrow & F & \longrightarrow & M_0 & \longrightarrow & 0
\end{array}$$

where the bottom row is the result of tensoring with F the exact sequence of R-modules $0 \to I \to R \to R_0 \to 0$. The isomorphisms in the induced commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \operatorname{Hom}_R(M_1,R_0) & \xrightarrow{\ \eth\ } & \operatorname{Ext}^1_R(M_0,R_0) & & \\
 & & & & {\scriptstyle\operatorname{Hom}_R(\nu,R_0)}\big\downarrow & & \| & & \\
 & & & & \operatorname{Hom}_R(R_1 \otimes_{R_0} M_0,R_0) & & \| & & \\
 & & & & \big\downarrow{\scriptstyle\cong} & & \| & & \\
\operatorname{Hom}_R(M_0,R_0) & \xrightarrow{\cong} & \operatorname{Hom}_R(F,R_0) & \longrightarrow & \operatorname{Hom}_R(I \otimes_R F,R_0) & \xrightarrow{\ \eth'\ } & \operatorname{Ext}^1_R(M_0,R_0) & \to & 0
\end{array}$$

are due to the isomorphism $F/IF \xrightarrow{\cong} M_0$. The exactness of the bottom row shows that the connecting map $\eth'$ is bijective. Furthermore, the following diagram

$$\begin{array}{ccc} \operatorname{Hom}_{R_0}(M_1,R_0) & \xrightarrow{\cong} & \operatorname{Hom}_R(M_1,R_0) \\ \Big\downarrow{\scriptstyle \operatorname{Hom}_{R_0}(\nu,R_0)} & & \Big\downarrow{\scriptstyle \operatorname{Hom}_R(\nu,R_0)} \\ \operatorname{Hom}_{R_0}(R_1\otimes_{R_0} M_0,R_0) & \xrightarrow{\cong} & \operatorname{Hom}_R(R_1\otimes_{R_0} M_0,R_0) \end{array}$$

commutes by functoriality. The horizontal arrows are isomorphisms because $R \to R_0$ is surjective. One gets the desired result by combining the last two diagrams □

3. Koszul modules

Theorem 1.1 shows that when $\mathfrak{m}$ has a Conca generator the asymptotic properties of arbitrary free resolutions are determined by those of resolutions of Koszul modules. In this section we turn to the problem of identifying and exhibiting such modules. The next result follows easily from work in the preceding section.

3.1. **Proposition.** *Let $(R,\mathfrak{m},k)$ be a Koszul local ring with $\mathfrak{m}^3=0$. Set*

$$e=\operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2)\,,\quad r=\operatorname{rank}_k(\mathfrak{m}^2)\,,\quad \textit{and}\quad \mathcal{E}=\operatorname{Ext}_R(k,k)\,.$$

For each finite R-module M with $\mathfrak{m}^2M=0$ the following are equivalent.

(i) *The R-module M is Koszul.*
(ii) *The Poincaré series of M over R is given by the formula*

$$P^R_M(t)=\frac{H_M(-t)}{1-et+rt^2}\,.$$

(iii) *The $\mathcal{E}$-module* $\operatorname{Ext}_R(M,k)$ *has projective dimension at most* 1.
(iv) *The map δ from Construction* 2.3 *is injective.*

Proof. By Remark 2.5, the resolution (2.4.3) yields (iv) $\iff$ (iii), and an equality

$$P^R_M(t)=\frac{H_M(-t)}{1-et+rt^2}+\left(1+\frac{1}{t}\right)\cdot\sum_{n=0}^{\infty}\operatorname{rank}_k(\mathcal{F}^n)\,t^n\,,$$

from which (iv) $\iff$ (ii) follows. The equality (2.4.1) establishes (iv) $\iff$ (i). □

Next we exhibit a substantial family of Koszul modules.

3.2. **Theorem.** *If $(R,\mathfrak{m},k)$ is a local ring and x is a Conca generator of $\mathfrak{m}$, then every finite R-module M annihilated by x is Koszul.*

For the proof we need a general change-of-rings result for the Koszul property.

3.3. **Lemma.** *Let $(R,\mathfrak{m},k)$ and $(R',\mathfrak{m}',k)$ be Koszul local rings, $\rho\colon R'\to R$ a Golod homomorphism, and M a finite R-module.*

If M is Koszul over R', then it is Koszul over R as well.

Proof. In view of [14, (6.1)], it suffices to show that the module M is ρ-Golod in the sense of Levin [16, (1.1)]; that is, the map $\operatorname{Tor}^\rho_n(M,k)$ is injective for each $n\ge 0$.

Fix a non-negative integer n. The exact sequence of R-modules

$$0\longrightarrow \mathfrak{m}M\xrightarrow{\iota} M\xrightarrow{\pi} M/\mathfrak{m}M\longrightarrow 0$$

induces an exact sequence of k-vector spaces

$$\operatorname{Tor}_n^{R'}(\mathfrak{m}M,k) \xrightarrow{\operatorname{Tor}_n^{R'}(\iota,k)} \operatorname{Tor}_n^{R'}(M,k) \xrightarrow{\operatorname{Tor}_n^{R'}(\pi,k)} \operatorname{Tor}_n^{R'}(M/\mathfrak{m}M,k)\,.$$

When M is Koszul over R' one has $\operatorname{Tor}_n^{R'}(\iota,k)=0$ by [21, (3.2)], and hence the map $\operatorname{Tor}_n^{R'}(\pi,k)$ is injective. It appears in a commutative diagram

$$\begin{array}{ccccc}
\operatorname{Tor}_n^{R'}(M,k) & \xrightarrow{\operatorname{Tor}_n^{R'}(\pi,k)} & \operatorname{Tor}_n^{R'}(M/\mathfrak{m}M,k) & \xrightarrow{\cong} & (M/\mathfrak{m}M)\otimes_k \operatorname{Tor}_n^{R'}(k,k) \\
\Big\downarrow{\scriptstyle \operatorname{Tor}_n^{\rho}(M,k)} & & \Big\downarrow{\scriptstyle \operatorname{Tor}_n^{\rho}(M/\mathfrak{m}M,k)} & & \Big\downarrow{\scriptstyle (M/\mathfrak{m}M)\otimes_k \operatorname{Tor}_n^{\rho}(k,k)} \\
\operatorname{Tor}_n^{R}(M,k) & \xrightarrow{\operatorname{Tor}_n^{R}(\pi,k)} & \operatorname{Tor}_n^{R}(M/\mathfrak{m}M,k) & \xrightarrow{\cong} & (M/\mathfrak{m}M)\otimes_k \operatorname{Tor}_n^{R}(k,k)
\end{array}$$

Since ρ is Golod, $\operatorname{Tor}_n^{\rho}(k,k)$ is injective by [1, (3.5)], hence so is $\operatorname{Tor}_n^{\rho}(M,k)$. □

The following explicit construction is also used in the proof of the theorem.

3.4. *Remark.* Choose a minimal generating set $\{x_1,\dots,x_e\}$ of $\mathfrak{m}$ as in 2.1, and a map $\pi\colon (P,\mathfrak{p},k)\to R$ as in Theorem 1.4. Pick elements $u_1,\dots,u_e$ in $\mathfrak{p}$ so that $\pi(u_i)=x_i$, and let X_i denote the leading form of u_i in the associated graded ring P^{g}. Thus, one has $P^{\mathsf{g}}=k[X_1,\dots,X_e]$, and the elements $X_1,\dots,X_e$ are algebraically independent over k. Let I be the ideal of P^{g} generated by

$$X_iX_j-\sum_{h=1}^{r}a_h^{ij}X_hX_e \quad \text{for} \quad 1\le i\le j\le e-1\,, \quad \text{and} \tag{3.4.1}$$

$$X_lX_e \quad \text{for} \quad r+1\le l\le e \tag{3.4.2}$$

where $r=\operatorname{rank}_k\mathfrak{m}^2$ and the elements $a_h^{ij}\in k$ are defined by formula (2.1.1).

Let $\overline{u}_i$ denote the image of u_i in the ring P^{g}/I. As $\overline{u}_e$ is a Conca generator of $(\overline{u}_1,\dots,\overline{u}_e)$, one has $(\overline{u}_1,\dots,\overline{u}_e)^3=0$, hence $\operatorname{rank}_k(P^{\mathsf{g}}/I)=1+e+r$. Since I is contained in the kernel of the map $\pi^{\mathsf{g}}\colon P^{\mathsf{g}}\to R^{\mathsf{g}}$ of graded k-algebras, it induces a surjection $P^{\mathsf{g}}/I\to R^{\mathsf{g}}$, which is bijective because $\operatorname{rank}_k R^{\mathsf{g}}=1+e+r$ holds.

Proof of Theorem 3.2. Let M be a finite R-module with $xM=0$.

To prove that M is Koszul it suffices to show that the graded R^{g}-module M^{g} has a linear free resolution; see [21, (2.3)] or [14, (1.5)]. By Remark 3.4, the ring R^{g} is local and the initial form $\overline{x}$ of x is a Conca generator of its maximal ideal. As $\overline{x}M^{\mathsf{g}}=0$ evidently holds, after changing notation we may assume that R is graded, $\mathfrak{m}$ has a Conca generator $x\in R_1$, $M_j=0$ for $j\neq 0,1$, and $M_1=R_1M_0$.

Remark 3.4 yields an isomorphism $R\cong k[X_1\dots,X_e]/I$, where I is generated by the quadratic forms in (3.4.1) and (3.4.2). Thus, there is a surjective homomorphism $\rho\colon R'\to R$ of graded k-algebras, where $R'=k[X_1\dots,X_e]/I'$ and I' is the ideal generated by X_e^2 and the forms in (3.4.1). The image of X_e is a Conca generator of the ideal $\mathfrak{m}'=(X_1\dots,X_e)R'$; in particular, R' is local with maximal ideal $\mathfrak{m}'$.

Theorem 1.1 shows that both rings R and R' are Koszul. Furthermore, one has an isomorphism $\operatorname{Ker}(\rho)\cong k(-2)^{e-r-1}$ of graded R'-modules, so $\operatorname{Ker}(\rho)$ has a 2-linear free resolution over R'. This implies that the homomorphism ρ is Golod, see [14, (5.8)]. Referring to Lemma 3.3 one sees that to finish the proof of the theorem it suffices to show that M has a linear free resolution over R'. Replacing R with R' and changing notation once more, we may also assume $\operatorname{rank}_k(R_2)=e-1$.

Next we prove $xR = (0 : x)_R$. The condition $x^2 = 0$ implies $xR \subseteq (0 : x)_R$. Equality holds because both ideals have the same rank: The exact sequences

$$0 \longrightarrow \mathfrak{m}^2 \longrightarrow xR \longrightarrow xR/\mathfrak{m}^2 \longrightarrow 0$$
$$0 \longrightarrow (0 : x)_R \longrightarrow R \longrightarrow xR \longrightarrow 0$$

yield $\operatorname{rank}_k xR = (e-1)+1 = e$ and $\operatorname{rank}_k(0 : x)_R = 2e - e = e$. The complex

$$\cdots \longrightarrow R(-2) \xrightarrow{x} R(-1) \xrightarrow{x} R \longrightarrow 0 \longrightarrow \cdots$$

is thus a minimal free resolution of the graded R-module R/xR, and hence R/xR is Koszul. As one has $xM = 0$, there is an exact sequence of graded R-modules

$$0 \longrightarrow k^q(-1) \longrightarrow (R/xR)^m \longrightarrow M \longrightarrow 0$$

with $m = \operatorname{rank}_k(M/\mathfrak{m}M)$. It induces for each pair (n, j) an exact sequence

$$\operatorname{Tor}_n^R((R/xR)^m, k)_j \longrightarrow \operatorname{Tor}_n^R(M, k)_j \longrightarrow \operatorname{Tor}_{n-1}^R(k^q, k)_{j-1}\,.$$

The vector spaces at both ends are zero when $n \neq j$, because both k and R/xR are Koszul over R. As a consequence, we get $\operatorname{Tor}_n^R(M, k)_j = 0$ for $n \neq j$, as desired. □

4. Gorenstein rings and related rings

In this section we study modules over local rings $(R, \mathfrak{m}, k)$ with $\mathfrak{m}^3 = 0$ and $\operatorname{rank}_k(\mathfrak{m}^2) \leq 1$, focusing on the Koszul property. We prove that, outside of an easily understood special case, every module has a Koszul syzygy. We completely describe the non-Koszul indecomposable modules when R is Gorenstein. An important finding is that for $e = \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$ the sequence $(b_n)_{n\geqslant 0}$ defined by

$$b_n = \begin{cases} n+1 & \text{when } e = 2\,; \\ \dfrac{1}{2^{n+1}} \displaystyle\sum_{j=0}^{\lfloor (n-1)/2 \rfloor} (e^2-4)^j e^{n-2j} & \text{when } e \geq 3\,, \end{cases} \tag{4.0.1}$$

provides numerical invariants for checking the Koszul property of R-modules.

4.1. **Theorem.** *Let $(R, \mathfrak{m}, k)$ be a local ring with $\mathfrak{m}^3 = 0$ and $\operatorname{rank}_k \mathfrak{m}^2 = 1$.*
For $e = \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$ and $s = \operatorname{rank}_k(0 : \mathfrak{m})$ the following are equivalent.

(i) *There is an inflation $R \to (R', \mathfrak{m}', k')$ such that $\mathfrak{m}'$ has a Conca generator.*
(ii) *R is Koszul.*
(iii) *$P_k^R(t) \cdot (1 - et + t^2) = 1$.*
(iv) *$\beta_n^R(k) = b_n$ for every integer $n \geq 0$.*
(v) *$s \leq e - 1$.*

The most useful property above is (i), as it has consequences for free resolutions of all R-modules; see Theorem 1.1 and Corollary 4.4. Thus, the main thrust of the theorem is that (i) follows from the easily verifiable condition (v).

The equivalence of conditions (ii), (iii), and a modified form of (v) is established by Fitzgerald in [9, (4.1)]. From the proof of that result we abstract the following statement, which it is not hard to verify directly.

4.2. *Remark.* A local ring $(R, \mathfrak{m}, k)$ with $\mathfrak{m}^3 = 0$ and $\operatorname{rank}_k \mathfrak{m}^2 = 1$ is isomorphic to a fiber product $S \times_k T$, where $(S, \mathfrak{s}, k)$ is a local ring with $\mathfrak{s}^2 = 0$ and $(T, \mathfrak{t}, k)$ is a Gorenstein local ring with $\mathfrak{t}^2 \neq 0$; see the proof that (1) implies (3) in [9, (4.1)].

Set $p = \operatorname{rank}_k(\mathfrak{s}/\mathfrak{s}^2)$ and $q = \operatorname{rank}_k(\mathfrak{t}/\mathfrak{t}^2)$ and note the evident relations

$$\operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2) = p + q \quad \text{and} \quad \operatorname{rank}_k(0 : \mathfrak{m}) = p + 1\,.$$

We also note an elementary observation, to be used more than once.

4.3. *Remark.* A Gorenstein local ring $(R, \mathfrak{m}, k)$ with $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$ satisfies

$$k \cong (0 : \mathfrak{m}) = \mathfrak{m}^2 = x\mathfrak{m} \quad \text{for every} \quad x \in \mathfrak{m} \smallsetminus \mathfrak{m}^2\,.$$

Indeed, the Gorenstein property of R implies $k \cong (0 : \mathfrak{m})$. As the hypotheses on $\mathfrak{m}$ mean $(0 : \mathfrak{m}) \supseteq \mathfrak{m}^2 \neq 0$, one concludes $(0 : \mathfrak{m}) = \mathfrak{m}^2$. Thus, one has either $x\mathfrak{m} = \mathfrak{m}^2$ or $x\mathfrak{m} = 0$, but the second option entails $x \in (0 : \mathfrak{m}) = \mathfrak{m}^2$, a contradiction.

Proof of Theorem 4.1. (i) $\implies$ (ii). This holds by Theorem 1.1.

(ii) $\implies$ (iii). This follows from Remark 1.9.

(iii) $\iff$ (iv). This is seen by decomposition into prime fractions.

In the rest of the proof we use the notation of Remark 4.2.

(iii) $\implies$ (v). The hypothesis and Remark 1.11 yields equalities

$$1 - et + t^2 = \frac{1}{P^R_k(t)} = \frac{1}{P^T_k(t)} + \frac{1}{P^S_k(t)} - 1 = \frac{1}{P^T_k(t)} - pt\,.$$

They imply $P^T_k(t)^{-1} = 1 - qt + t^2$. This rules out the case $q = 1$, because the local ring T then has embedding dimension 1 so one has $P^T_k(t)^{-1} = 1 - t$. Thus $s = p + 1 = e - q + 1 \leq e - 1$.

(v) $\implies$ (i). There exists a local ring $(R', \mathfrak{m}', k')$ with k' algebraically closed and an inflation $R \to R'$, see [10, Chapter 0, 10.3.1]. One then has $\mathfrak{m}'^3 = 0$ and $\operatorname{rank}_{k'} \mathfrak{m}'^2 = 1$; also, $\operatorname{rank}_{k'}(\mathfrak{m}'/\mathfrak{m}'^2) = e$ and $\operatorname{rank}_{k'}(0 : \mathfrak{m}') = s$ hold. Thus, it suffices to prove that $\mathfrak{m}$ has a Conca generator when k is algebraically closed.

The Gorenstein ring T in the decomposition $R = S \times_k T$ has $q = e - p = e - (s - 1) \geq 2$. A Conca generator of $\mathfrak{t}$ clearly also is a Conca generator of $\mathfrak{m}$. Thus, we may further assume that R is Gorenstein with $e \geq 2$; this implies $\mathfrak{m}^2 \neq 0$.

Remark 4.3 shows that for each $x \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ one has $x\mathfrak{m} = \mathfrak{m}^2$. On the other hand, by [5, Lem. 3] one can choose an element x as above, so that $\overline{x} \in \mathfrak{m}/\mathfrak{m}^2 = R^{\mathfrak{g}}_1$ satisfies $\overline{x}^2 = 0$. This yields $x^2 \in \mathfrak{m}^3 = 0$, so x is a Conca generator of $\mathfrak{m}$. □

Next we recover and extend Sjödin's description [20] of Poincaré series of modules over Gorenstein local rings with $\mathfrak{m}^3 = 0$. Note that the second case below differs from the other two, as all series have a common denominator *different from* $H_R(-t)$.

4.4. **Corollary.** *Let* $(R, \mathfrak{m}, k)$ *be a local ring with* $\mathfrak{m}^3 = 0$ *and* $\operatorname{rank}_k \mathfrak{m}^2 \leq 1$.

The numbers $e = \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$ *and* $s = \operatorname{rank}_k(0 : \mathfrak{m})$ *then satisfy* $s \leq e$, *and for every finite* R*-module* M *one has*

$$\deg\big(P^R_M(t)\cdot(1 - et)\big) \leq 1 \quad \textit{when } \mathfrak{m}^2 = 0.$$

$$\deg\big(P^R_M(t)\cdot(1 - et)\big) \leq 2 \quad \textit{when } \operatorname{rank}_k \mathfrak{m}^2 = 1 \textit{ and } s = e.$$

$$\deg\big(P^R_M(t)\cdot(1 - et + t^2)\big) < \infty \quad \textit{when } \operatorname{rank}_k \mathfrak{m}^2 = 1 \textit{ and } s < e.$$

Proof. When $\mathfrak{m}^2 = 0$ one has $\Omega^R_1(M) \cong k^a$ for some $a \geq 0$, and hence

$$P^R_M(t) - \beta^R_0(M) = P^R_{k^a}(t)\cdot t = \frac{a}{1 - et}\cdot t$$

When $\operatorname{rank}_k \mathfrak{m}^2 = 1$ and $s = e$ Remark 4.2 yields $R = S \times_k T$, where the local ring $(T, \mathfrak{t}, k)$ has $\operatorname{rank}_k(\mathfrak{t}/\mathfrak{t}^2) = 1$. Thus, the maximal ideal $\mathfrak{t}$ is principal, so every indecomposable T-module is isomorphic to k, $\mathfrak{t}$, or T.

By [8, Rem. 3] one has $\Omega_2^R(M) = K \oplus L$, where K is an S-module and L is a T-module. The inclusion $\Omega_2^R(M) \subseteq \mathfrak{m}F_1$, where F_1 is a free R-module, gives $\mathfrak{s}K \subseteq \mathfrak{s}^2F_1 = 0$ and $\mathfrak{t}^2L \subseteq \mathfrak{t}^3F_1 = 0$. Thus, there exist integers $a, b, c \geq 0$ and isomorphisms $K \cong k^a$ and $L \cong k^b \oplus \mathfrak{t}^c$ of S-modules and T-modules, respectively.

For $d = e - 1$ the discussion above leads to the first and third equalities below:

$$\begin{aligned} P_M^R(t) - \beta_0^R(M) - \beta_1^R(M) \cdot t &= P_K^R(t) \cdot t^2 + P_L^R(t) \cdot t^2 \\ &= \frac{P_K^S(t) \cdot P_k^T(t) + P_k^S(t) \cdot P_L^T(t)}{P_k^S(t) + P_k^T(t) - P_k^S(t) \cdot P_k^T(t)} \cdot t^2 \\ &= \frac{\dfrac{a}{1-dt} \cdot \dfrac{1}{1-t} + \dfrac{1}{1-dt} \cdot \dfrac{b+c}{1-t}}{\dfrac{1}{1-dt} + \dfrac{1}{1-t} - \dfrac{1}{1-dt} \cdot \dfrac{1}{1-t}} \cdot t^2 \\ &= \frac{a+b+c}{1-et} \cdot t^2 \end{aligned}$$

The second equality comes from the change-of-rings formula in Remark 1.11.

When $\operatorname{rank}_k \mathfrak{m}^2 = 1$ and $s < e$ the result follows from Theorems 4.1 and 1.1. □

It follows from work of Conca that, under additional hypotheses, certain conditions of Theorem 4.1 remain valid for larger values of $\operatorname{rank}_k \mathfrak{m}^2$.

4.5. *Remark.* Let $(R, \mathfrak{m}, k)$ be a local ring with $\mathfrak{m}^3 = 0$, field k of characteristic different from 2, and k-algebra R^{g} defined by quadratic relations.

When $\operatorname{rank}_k \mathfrak{m}^2 = 2$ holds there is an inflation $R \to (R', \mathfrak{m}', k')$, such that $\mathfrak{m}'$ has a Conca generator; when $\operatorname{rank}_k \mathfrak{m}^2 = 3$ holds the ring R is Koszul.

Indeed, as in the proof of (v) $\implies$ (i) in Theorem 4.1 one may reduce to the case when R equals R^{g} and k is algebraically closed. If $\operatorname{rank}_k R_2 = 2$, then the proof of [5, Prop. 6] shows that the ideal (R_1) has a Conca generator. If $\operatorname{rank}_k R_2 = 3$, then the ring R is G-quadratic by [6, Thm. 1.1(2)]; in particular, R is a Koszul ring.

We turn to Koszul modules over a Gorenstein local ring $(R, \mathfrak{m}, k)$ with $\mathfrak{m}^3 = 0$.

Recall that when $\operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2) = 1$ the ring R has finite representation type: When $\mathfrak{m}^2 = 0$ the indecomposable R-modules are k and R, and both are Koszul. When $\mathfrak{m}^2 \neq 0$ the indecomposable modules are k, $\mathfrak{m}$, and R; only R is Koszul.

When $\operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2) = 2$, the field k is algebraically closed, and $\operatorname{char}(k)R = 0$, one has $R \cong k[X_1, X_2]/(X_1^2, X_2^2)$. The ring R has tame representation type, and its indecomposable modules are described by Kronecker's classification of pairs of commuting matrices. From this description one can deduce that the negative syzygies of k are the only non-Koszul indecomposable R-modules.

We prove that the latter property holds for all Gorenstein local rings with $\mathfrak{m}^3 = 0$ and $e \geq 2$; this may surprise, as their representation theory is wild when $e \geq 3$.

4.6. **Theorem.** *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring with $\mathfrak{m}^3 = 0$, and set*

$$\Omega_{-n}^R(k) = \operatorname{Hom}_R(\Omega_n^R(k), R) \quad \textit{for each} \quad n \geq 1\,.$$

Set $e = \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$, assume $e \geq 2$ holds, and for $n \geq 0$ define b_n by (4.0.1).

A finite R-module M is Koszul if and only if it has no direct summand isomorphic to $\Omega_{-i}^R(k)$ with $i \geq 1$; when M is indecomposable the following are equivalent.

(i) *M is not Koszul.*

(ii) *$M \cong \Omega_{-i}^R(k)$ for some $i \geq 1$.*

(iii) $H_M(t) = b_{i-1} + b_i\, t$.

Proof. It suffices to verify the equivalence of conditions (i) through (iii) when M is an indecomposable, non-free, finite R-module. These properties imply $\mathfrak{m}^2M = 0$.

Indeed, by Remark 4.3 one has $\mathfrak{m}^2 = (0 : \mathfrak{m}) \cong k$. Let s be a generator of $\mathfrak{m}^2$ and $y \in M$ an element with $sy \neq 0$. The R-linear map $R \to M$ sending 1 to y is injective since it is injective on $(0 : \mathfrak{m})$. It is then split, for R is self-injective, and so M has a direct summand isomorphic to R; a contradiction.

(i) $\implies$ (ii). Proposition 3.1 yields $P^R_M(t) \neq H_M(-t)/H_R(-t)$. By a result of Lescot, see [15, 3.4(1)], then k is isomorphic to a direct summand of $\Omega^R_i(M)$ for some $i \geq 1$. As $\Omega^R_i(M)$ is indecomposable along with M, see [13, 1.3], one gets $\Omega^R_i(M) \cong k$, hence $M \cong \Omega^R_{-i}(k)$ by Matlis duality.

(ii) $\implies$ (iii). A minimal free resolution F of M yields an exact sequence

$$0 \longrightarrow k \longrightarrow F_{i-1} \longrightarrow F_{i-2} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0\,. \tag{4.6.1}$$

Set $G_n = \operatorname{Hom}_R(F_{i-1-n}, R)$. Applying $\operatorname{Hom}_R(-, R)$ one gets an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(M, R) \longrightarrow G_{i-1} \longrightarrow G_{i-2} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow k \longrightarrow 0$$

with $\partial(G_n) \subseteq \mathfrak{m}G_{n-1}$ for $n = 0, \dots, i-1$. From it and Theorem 4.1 one obtains

$$\operatorname{rank}_k(M/\mathfrak{m}M) = \operatorname{rank}_R(F_0) = \operatorname{rank}_R(G_{i-1}) = b_{i-1}\,.$$

Note that $\mathfrak{m}M \neq 0$; else $b_i = 1$ for some $i \geq 1$, which cannot be the case.

Since $\mathfrak{m}^2M = 0$, one has that $\mathfrak{m}M \subseteq (0 : \mathfrak{m})_M$. The reverse inclusion holds because the composed map $(0 : \mathfrak{m})_M \to M \to M/\mathfrak{m}M$ has to be zero, as M is indecomposable. Thus, $\mathfrak{m}M = (0 : \mathfrak{m})_M$, which gives the first equality below; Matlis duality gives the second one, and (4.6.1) the third:

$$\operatorname{rank}_k(\mathfrak{m}M) = \operatorname{rank}_k((0 : \mathfrak{m})_M) = \operatorname{rank}_k(k \otimes_R \Omega^R_i(k)) = \operatorname{rank}_R(G_i) = b_i\,.$$

(iii) $\implies$ (i). Assuming that M is Koszul, from Proposition 3.1 one gets

$$P^R_M(t) = (b_{i-1} - b_i t) \cdot P^R_k(t) = (b_{i-1} - b_i t) \cdot \sum_{n=0}^{\infty} b_n t^n\,,$$

hence $\beta^R_i(M) = b_{i-1}b_i - b_i b_{i-1} = 0$. Thus, M has finite projective dimension. It is free because R is artinian, contradicting the hypotheses $\mathfrak{m}^2M = 0 \neq M$. □

4.7. **Corollary.** *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring with $\mathfrak{m}^3 = 0$.*

Set $e = \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$. If M is an indecomposable finite R-module such that an inequality $\operatorname{rank}_k(\mathfrak{m}M) \leq (e-1)\operatorname{rank}_k(M/\mathfrak{m}M)$ holds, then M is Koszul.

Proof. The equivalence of (iii) and (iv) in Theorem 4.1 means that $(b_n)_{n\geq 0}$ satisfies the recurrence relation $b_{n+1} = eb_n - b_{n-1}$ for $n \geq 2$, with $b_0 = 1$ and $b_1 = e$. It implies an inequality $b_n > (e-1)b_{n-1}$ for each $n \geq 0$, so M fails test (iii) of Theorem 4.6. □

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