EXTENSIONS OF A DUALIZING COMPLEX BY ITS RING: COMMUTATIVE VERSIONS OF A CONJECTURE OF TACHIKAWA

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Dedicated to Wolmer Vasconcelos on the occasion of his 65th birthday

Abstract. Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring with dualizing complex D^R, normalized by $\operatorname{Ext}_R^{\operatorname{depth}(R)}(k, D^R) \cong k$. Partly motivated by a long standing conjecture of Tachikawa on (not necessarily commutative) k-algebras of finite rank, we conjecture that if $\operatorname{Ext}_R^n(D^R, R) = 0$ for all $n > 0$, then R is Gorenstein, and prove this in several significant cases.

Introduction

Let $(R, \mathfrak{m}, k)$ be a *local ring*, that is, a commutative noetherian ring R with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. We write $\operatorname{edim} R$ for the minimal number of generators of $\mathfrak{m}$ and set $\operatorname{codepth} R = \operatorname{edim} R - \operatorname{depth} R$.

We assume that R has a dualizing complex, and let D^R denote such a complex shifted so that $\mathrm{H}_i(D^R) = 0$ for $i < 0$ and $\mathrm{H}_0(D^R) \neq 0$; see Section 1 for more details on dualizing complexes. The ring R is Cohen-Macaulay (respectively, Gorenstein) if and only if D^R can be taken to be a dualizing module for R (respectively, to be the R-module R). Thus, when R is Gorenstein, $\operatorname{Ext}_R^i(D^R, R) = 0$ holds trivially for all $i > 0$. We prove that vanishing of such Ext groups often implies that R is Gorenstein, as follows:

Section 2. R is generically Gorenstein (e.g., R is reduced), and $\operatorname{Ext}_R^i(D^R, R) = 0$ for all $i \in [1, \dim R]$.

Section 3. R is a homomorphic image of a generically Gorenstein ring with dualizing complex, and $\operatorname{Ext}_R^i(D^R, R) = 0$ for all $i \in [1, \dim R + 1]$.

Section 4. R is Cohen-Macaulay, in the linkage class of a complete intersection (e.g., $\operatorname{codepth} R \leq 2$), and $\operatorname{Ext}_R^i(D^R, R) = 0$ for all $i \in [1, \dim R + 1]$.

Section 5. R contains a regular sequence $\boldsymbol{f}$ with the property that $\mathfrak{m}^3 \subseteq (\boldsymbol{f})$, and $\operatorname{Ext}_R^i(D^R, R) = 0$ for all $i \in [1, \dim R + 1]$.

Section 6. R is the special fiber of a finite flat local homomorphism that possesses a Gorenstein fiber, and $\operatorname{Ext}_R^1(D^R, R) = 0$.

Section 7. $\operatorname{codepth} R \leq 3$ and $\operatorname{Ext}_R^i(D^R, R) = 0$ for all $i \gg 0$.

Section 8. R is Golod and $\operatorname{Ext}_R^i(D^R, R) = 0$ for all $i \gg 0$.

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An effort to make sense of the disparate hypotheses leads us to pose the

Question. Does the vanishing of $\operatorname{Ext}^i_R(D^R, R) = 0$ for $(\dim R + 1)$ consecutive positive values of i imply that R is Gorenstein?

A result of Foxby [10], see Section 1, implies that if $\operatorname{Ext}^i_R(D^R, R)$ vanishes for $i = 1, \dots, \dim R$, then R is Cohen-Macaulay, so this hypothesis is implicit in the

Conjecture. If $\operatorname{Ext}^i_R(D^R, R) = 0$ for all $i > 0$, then R is Gorenstein.

This is an analog of a 30 years old conjecture of Tachikawa [26] on (possibly non-commutative) algebras of finite rank over a field. In fact, if the hypotheses of both conjectures apply, then they are equivalent. The result of Section 5 is obtained by transposing the proof, due to Asashiba and Hoshino [1], [2], of one case of Tachikawa's conjecture. On the other hand, our results prove that conjecture in new cases. Background material and more details on Tachikawa's conjecture are given in Section 9.

Our arguments use a substantial amount of homological algebra for complexes of modules. Appendix A contains a synopsis of the basic constructions. In Appendix B we have collected miscellaneous results on complexes that play central roles in various proofs, but do not draw upon the framework of noetherian local algebra.

1. Dualizing complexes

Let $(R, \mathfrak{m}, k)$ be a local ring. In this section we collect basic properties of dualizing complexes. A comprehensive treatment can be found in Foxby's notes [10]. We refer to Appendix A for basic notation regarding complexes.

A complex of R-modules is said to be *dualizing* if it has finite homology and there is an integer d such that $\operatorname{Ext}^d_R(k, D) \cong k$ and $\operatorname{Ext}^i_R(k, D) = 0$ for $i \neq d$. For completeness, we recall that this is the case if and only if R is a homomorphic image of a Gorenstein ring, where the "if" part is a deep recent result of Kawasaki [27]. In this paper, we always assume that R has a dualizing complex.

1.1. Let D^R denote a dualizing complex with $\inf \mathrm{H}(D^R) = 0$, and set $d = \operatorname{depth} R$.

1.1.1. Every dualizing complex is quasi-isomorpic to a shift of D^R, cf. [10, 15.14].

1.1.2. $\sup \mathrm{H}(D^R) = \dim R - d$, cf. [10, 15.18].

1.1.3. $\operatorname{Ext}^i_R(k, D^R) = 0$ for $i \neq d$ and $\operatorname{Ext}^d_R(k, D^R) \cong k$, cf. [10, 15.18].

1.1.4. For every $\mathfrak{p} \in \operatorname{Spec} R$ and for $n = \operatorname{depth} R - \operatorname{depth} R_{\mathfrak{p}} - \dim(R/\mathfrak{p})$ there is a quasi-isomorphism $\Sigma^n(D^R)_{\mathfrak{p}} \simeq D^{R_{\mathfrak{p}}}$ of complexes of $R_{\mathfrak{p}}$-modules, cf. [10, 15.17].

The next result slightly extends an observation by the referee.

1.2. Proposition. *If $\mathfrak{p}$ is an associated prime ideal of R, then*

$$\operatorname{Ext}^i_R(D^R, R) \neq 0 \quad \textit{for} \quad i = \dim(R/\mathfrak{p}) - \operatorname{depth} R$$

Proof. Since D^R has finite homology, $\operatorname{Ext}_R(D^R, R)$ localizes, cf. [10, 6.47]. This justifies the first isomorphism below:

$$\begin{aligned}\operatorname{Ext}^i_R(D^R, R)_{\mathfrak{p}} &\cong \operatorname{Ext}^i_{R_{\mathfrak{p}}}\left((D^R)_{\mathfrak{p}}, R_{\mathfrak{p}}\right) \cong \operatorname{Ext}^i_{R_{\mathfrak{p}}}\left(\Sigma^i(D^{R_{\mathfrak{p}}}), R_{\mathfrak{p}}\right) \\ &\cong \operatorname{Ext}^0_{R_{\mathfrak{p}}}(D^{R_{\mathfrak{p}}}, R_{\mathfrak{p}})\end{aligned}$$

The second isomorphism comes from 1.1.4 and the last one is due to the shift. Now $\inf \mathrm{H}(D^{R_\mathfrak{p}}) = 0$ by 1.1.2, and hence it follows from [10, 6.42(3)] that

$$\mathrm{Ext}^0_{R_\mathfrak{p}}(D^{R_\mathfrak{p}}, R_\mathfrak{p}) \cong \mathrm{Hom}_R(\mathrm{H}_0(D^{R_\mathfrak{p}}), R_\mathfrak{p})$$

Since $\mathrm{H}_0(D^{R_\mathfrak{p}}) \neq 0$ by 1.1.2 and $\operatorname{depth} R_\mathfrak{p} = 0$, the module $\mathrm{Hom}_R(\mathrm{H}_0(D^{R_\mathfrak{p}}), R_\mathfrak{p})$ is nonzero, so we conclude $\mathrm{Ext}^i_R(D^R, R)_\mathfrak{p} \neq 0$, and hence $\mathrm{Ext}^i_R(D^R, R) \neq 0$. □

1.3. **Corollary.** *If $\mathrm{Ext}^i_R(D^R, R) = 0$ for all $i \in [1, \dim R]$, then the ring R is Cohen-Macaulay.*

Proof. Choose $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim(R/\mathfrak{p}) = \dim R$. Note that $\mathfrak{p}$ is associated to R and that $\dim(R/\mathfrak{p}) - \operatorname{depth} R = i$, so $\mathrm{Ext}^i_R(D^R, R) \neq 0$ by the proposition. In view of our hypothesis, this implies $i = 0$, that is, R is Cohen-Macaulay. □

A special case of the property 1.1.2 reads:

1.4. The ring R is Cohen-Macaulay if and only if $\mathrm{H}_i(D^R) = 0$ for all $i \neq 0$. When this is the case, D^R can be taken to be a module, called a *dualizing module* of R.

For the properties of dualizing modules listed below we refer to [7, §3.3].

1.5. Let R be a Cohen-Macaulay local ring.

1.5.1. The ring R has a dualizing module D if and only if there exists a finite homomorphism of rings $Q \to R$ with a Gorenstein local ring Q; when this is the case, every dualizing module is isomorphic to $\mathrm{Ext}^{\dim Q - \dim R}_Q(R, Q)$.

1.5.2. $\mathrm{id}_R D = \dim R$.

1.5.3. $\mathrm{rank}_k(D/\mathfrak{m}D) = \mathrm{rank}_k(\mathrm{Ext}^{\dim R}_R(k, R))$.

1.5.4. $\mathrm{Ass}_R D = \mathrm{Ass}\, R$.

1.5.5. For each $\mathfrak{p} \in \operatorname{Spec} R$ the module $D_\mathfrak{p}$ is dualizing for $R_\mathfrak{p}$.

1.5.6. Each R-regular element f is also D-regular, and D/fD is dualizing for $R/(f)$.

1.5.7. $D \cong R$ if and only if the ring R is Gorenstein.

2. Generically Gorenstein rings

A commutative ring R is said to be *generically Gorenstein* if the ring $R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Ass}\, R$. Our purpose in this section is to prove

2.1. **Theorem.** *Let R be a generically Gorenstein local ring.*
If $\mathrm{Ext}^i_R(D^R, R) = 0$ for all $i \in [1, \dim R]$, then R is Gorenstein.

Remark. Another proof of Theorem 2.1 was independently obtained by Huneke and Hanes [11, 2.2] when R is Cohen-Macaulay. The essential input comes from [11, 2.1], which implies that $D^R \otimes_R D^R$ is a maximal Cohen-Macaulay module. The same conclusion can be obtained from Corollary B.4(3), that has weaker hypotheses.

We start with a criterion for a module to be free of rank 1. With a view towards later applications, we work in greater generality than needed here.

2.2. **Lemma.** *Let $P \to Q$ be a finite homomorphism of commutative rings, and let N be a finite Q-module. If P is local, $\mathrm{Ass}_P(N \otimes_Q N) \cup \mathrm{Ass}_P\, Q \subseteq \mathrm{Ass}\, P$, and $N_\mathfrak{p} \cong Q_\mathfrak{p}$ as $Q_\mathfrak{p}$-modules for every prime ideal $\mathfrak{p} \in \mathrm{Ass}\, P$, then $N \cong Q$.*

Proof. The second symmetric power $\mathsf{S}^2_Q(N)$ appears in a canonical epimorphism

$$\pi^N_Q \colon N \otimes_Q N \longrightarrow \mathsf{S}^2_Q(N)$$

For each $\mathfrak{p} \in \operatorname{Ass} P$ the homomorphism $(\pi^N_Q)_{\mathfrak{p}}$ factors as a composition

$$(N \otimes_Q N)_{\mathfrak{p}} \cong N_{\mathfrak{p}} \otimes_{Q_{\mathfrak{p}}} N_{\mathfrak{p}} \xrightarrow{\pi^{N_{\mathfrak{p}}}_{Q_{\mathfrak{p}}}} \mathsf{S}^2_{Q_{\mathfrak{p}}}(N_{\mathfrak{p}}) \cong (\mathsf{S}^2_Q(N))_{\mathfrak{p}}$$

with canonical isomorphisms. As the $Q_{\mathfrak{p}}$-module $N_{\mathfrak{p}}$ is cyclic, the map $\pi^{N_{\mathfrak{p}}}_{Q_{\mathfrak{p}}}$ is bijective, so $(\pi^N_Q)_{\mathfrak{p}}$ is an isomorphism, hence $(\operatorname{Ker}(\pi^N_Q))_{\mathfrak{p}} = 0$. The inclusions

$$\operatorname{Ass}_P(\operatorname{Ker}(\pi^N_Q)) \subseteq \operatorname{Ass}_P(N \otimes_Q N) \subseteq \operatorname{Ass} P$$

now imply $\operatorname{Ker}(\pi^N_Q) = 0$, so π^N_Q is bijective.

Let $\mathfrak{n}$ be a maximal ideal of Q and set $r = \operatorname{rank}_{Q/\mathfrak{n}}(N/\mathfrak{n}N)$. Assuming $r = 0$, and localizing at $\mathfrak{n}$, we get $N_{\mathfrak{n}} = 0$ by Nakayama's Lemma. Since Q is finite as a P-module, $\mathfrak{n} \cap P$ is the maximal ideal of P, hence $\mathfrak{n}$ contains $\mathfrak{p} \in \operatorname{Ass} P$. Thus, $N_{\mathfrak{p}}$ is a localization of $N_{\mathfrak{n}}$, hence $N_{\mathfrak{p}} = 0$, contradicting our hypothesis. We conclude that $r \geq 1$. By naturality, the map π^N_Q induces an isomorphism

$$(N/\mathfrak{n}N) \otimes_{Q/\mathfrak{n}} (N/\mathfrak{n}N) \cong \mathsf{S}^2_{Q/\mathfrak{n}}(N/\mathfrak{n}N)$$

of vector spaces over $Q/\mathfrak{n}$. Comparing ranks, we get $r = 1$.

The ring Q, being a finite module over a local ring, is semilocal. Thus, if $\mathfrak{r}$ is its Jacobson radical, then $Q/\mathfrak{r}$ is a finite product of fields. By what we have just proved, there is an isomorphism $Q/\mathfrak{r}Q \cong N/\mathfrak{r}N$ of $Q/\mathfrak{r}Q$-modules. Nakayama's Lemma implies that the Q-module N is cyclic. Choose an epimorphism $\varkappa \colon Q \to N$. For each $\mathfrak{p} \in \operatorname{Ass} P$ we have $N_{\mathfrak{p}} \cong Q_{\mathfrak{p}}$ by hypothesis, so $\varkappa_{\mathfrak{p}}$ is bijective, hence $(\operatorname{Ker}(\varkappa))_{\mathfrak{p}} = 0$. Since $\operatorname{Ass}_P(\operatorname{Ker}(\varkappa))$ is contained in $\operatorname{Ass}_P Q$, and by hypothesis the latter set lies in $\operatorname{Ass} P$, it follows that $\operatorname{Ker}(\varkappa) = 0$, so $\varkappa$ is an isomorphism. □

Proof of Theorem 2.1. By Corollary 1.3 the ring R is Cohen Macaulay, so in view of 1.4 we may assume that D^R is an R-module, D. Since $\operatorname{id}_R D = \dim R$, cf. 1.5.2, Corollary B.4 yields an isomorphism $D \otimes_R D \cong \operatorname{Hom}_R(D^*, D)$. This gives the first equality in the chain

$$\operatorname{Ass}_R(D \otimes_R D) = \operatorname{Ass}_R \operatorname{Hom}_R(D^*, D) = \operatorname{Supp}_R(D^*) \cap \operatorname{Ass}_R D \subseteq \operatorname{Ass} R$$

where the second equality comes from a classical expression for the associator of a module of homomorphisms, and the last inclusion from property 1.5.4.

As $R_{\mathfrak{p}}$ is Gorenstein for each $\mathfrak{p} \in \operatorname{Ass} R$, we have $D_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ by 1.5.5 and 1.5.7. Lemma 2.2 now yields $D \cong R$, so R is Gorenstein by 1.5.7. □

3. Homomorphic images of generically Gorenstein rings

In this section, Q denotes a generically Gorenstein homomorphic image of a Gorenstein local ring.

Admitting slightly more vanishing, we extend the scope of Theorem 2.1.

3.1. **Theorem.** *Assume $R \cong Q/(\boldsymbol{g})$, where $\boldsymbol{g}$ is a Q-regular set.*
If $\operatorname{Ext}^i_R(D^R, R) = 0$ for all $i \in [1, \dim R + 1]$, then R is Gorenstein.

We start with a lemma that grew out of a discussion with Bernd Ulrich.

3.2. **Lemma.** *If Q is Cohen-Macaulay, then $R \cong P/(g)$, where P is a generically Gorenstein, Cohen-Macaulay homomorphic image of a Gorenstein local ring and g is P-regular.*

Proof. Being a homomorphic image of a Gorenstein ring, Q contains an ideal $\mathfrak{d}$ such that for every $\mathfrak{q} \in \operatorname{Spec} Q$, the ring $Q_\mathfrak{q}$ is Gorenstein if and only if $\mathfrak{q} \supseteq \mathfrak{d}$. Let $\mathfrak{n}$ denote the maximal ideal of Q and set $X = \{\mathfrak{q} \in \operatorname{Spec} Q \mid \operatorname{height} \mathfrak{q} = 1 \text{ and } \mathfrak{q} \supseteq \mathfrak{d}\}$. If $\operatorname{card}(\boldsymbol{g}) = s > 1$ then X is finite, so pick $g_1 \in (\boldsymbol{g}) \smallsetminus \big(\mathfrak{n}(\boldsymbol{g}) \cup \bigcup_{\mathfrak{q}\in X} \mathfrak{q}\big)$ using prime avoidance. The ideal $\mathfrak{d}' = \mathfrak{d} + (g_1)/(g_1)$ defines the non-Gorenstein locus of $Q' = Q/(g_1)$ and has $\operatorname{height} \mathfrak{d}' > 0$, so Q' is generically Gorenstein. Extend g_1 to a minimal set $g_1, \dots, g_s$ generating $(\boldsymbol{g})$. This set is Q-regular. Thus, Q' is a Cohen-Macaulay homomorphic image of a Gorenstein local ring, the images of $g_2, \dots, g_s$ in Q' form a Q'-regular set $\boldsymbol{g}'$, and $Q'/(\boldsymbol{g}') \cong R$. To obtain the desired result, iterate the procedure $s-1$ times. □

Next we record an easy result on change of rings.

3.3. **Lemma.** *Let P be a local noetherian ring, let M, N be finite P-modules, let $g \in P$ be a $P \oplus M \oplus N$-regular element, and let n be a positive integer.*

(1) *If $\operatorname{Ext}^n_{P/(g)}(M/gM, N/gN) = 0$, then $\operatorname{Ext}^n_P(M,N) = 0$.*
(2) *If $\operatorname{Ext}^i_P(M,N) = 0$ for $i = n, n+1$, then $\operatorname{Ext}^n_{P/(g)}(M/gM, N/gN) = 0$.*

Proof. The canonical isomorphisms $\operatorname{Ext}^{i+1}_P(M/gM, N) \cong \operatorname{Ext}^i_{P/(g)}(M/gM, N/gN)$, holding for all $i \geq 0$, show that the exact sequence of P-modules

$$0 \to M \xrightarrow{g} M \to M/gM \to 0$$

yields for every $i \geq 0$ an exact sequence

$$\operatorname{Ext}^i_P(M,N) \xrightarrow{g} \operatorname{Ext}^i_P(M,N) \longrightarrow \operatorname{Ext}^i_{P/(g)}(M/gM, N/gN) \longrightarrow \operatorname{Ext}^{i+1}_P(M,N)$$

This sequence establishes (2). It also shows that the hypothesis of (1) implies $g \operatorname{Ext}^i_P(M,N) = 0$, so Nakayama's Lemma yields the desired assertion. □

Proof of Theorem 3.1. By Corollary 1.3 the ring R is Cohen Macaulay, so Lemma 3.2 yields $R \cong P/(g)$, with g a P-regular element. From Lemma 3.3(1) and 1.5.6 we then get $\operatorname{Ext}^i_P(D^P, P) = 0$ for all $i \in [1, \dim P]$. Thus, the ring P is Gorenstein by Theorem 2.1, hence so is R. □

4. Well linked Cohen-Macaulay rings

In this section we describe a class of rings to which Theorem 3.1 applies. They are constructed by means of linkage, so we start by recalling some terminology.

Let Q be a Gorenstein local ring and $\mathfrak{a}$ an ideal in Q such that the ring $Q/\mathfrak{a}$ is Cohen-Macaulay. An ideal $\mathfrak{b}$ of Q is said to be *linked* to $\mathfrak{a}$ if $\mathfrak{b} = (\boldsymbol{g}) : \mathfrak{a}$ and $\mathfrak{a} = (\boldsymbol{g}) : \mathfrak{b}$ for some Q-regular sequence $\boldsymbol{g}$ in $\mathfrak{a} \cap \mathfrak{b}$; when this is the case, we write $\mathfrak{b} \sim \mathfrak{a}$; the ring $Q/\mathfrak{b}$ is Cohen-Macaulay, cf. Peskine and Szpiro [21, 1.3].

We say that an ideal $\mathfrak{b}$ is *generically complete intersection*, if for every $\mathfrak{q} \in \operatorname{Ass}_Q(Q/\mathfrak{b})$ the ideal $\mathfrak{b}_\mathfrak{q}$ in $Q_\mathfrak{q}$ is generated by a $Q_\mathfrak{q}$-regular sequence.

4.1. **Theorem.** *Let R be a Cohen-Macaulay local ring of the form $Q/\mathfrak{a}$, where Q is a Gorenstein local ring and $\mathfrak{a}$ is an ideal for which there is a sequence of links $\mathfrak{a} \sim \mathfrak{b}_1 \sim \cdots \sim \mathfrak{b}_s \sim \mathfrak{b}$ with $\mathfrak{b}$ a generically complete intersection ideal.*

If $\operatorname{Ext}^i_R(D^R, R) = 0$ for all $i \in [1, \dim R + 1]$, then R is Gorenstein.

The proof uses Theorem 3.1, as well as work of Huneke and Ulrich [16], [17].

Proof. The construction of generic links of $\mathfrak{b}$ in [17, 2.17(a)] provides a prime ideal $\mathfrak{r}$ in some polynomial ring $Q[\boldsymbol{x}]$, an ideal $\mathfrak{c}$ in the local ring $Q[\boldsymbol{x}]_{\mathfrak{r}}$, a regular set $\boldsymbol{g}'$ in $Q' = Q[\boldsymbol{x}]_{\mathfrak{r}}/\mathfrak{c}$, and an isomorphism $R \cong Q'/(\boldsymbol{g}')$. The ring Q' is Cohen-Macaulay because R is. By [16, 2.9(b)] the ideal $\mathfrak{c}$ is generically complete intersection along with $\mathfrak{b}$. This implies that Q' is generically Gorenstein. It is also a residue ring of the Gorenstein ring $Q[\boldsymbol{x}]_{\mathfrak{r}}$, so Theorem 3.1 implies that the ring R is Gorenstein. □

The theorem covers the case when R is in the *linkage class of a complete intersection*, that is, in some Cohen presentation $\widehat{R} \cong Q/\mathfrak{a}$ with Q a regular local ring, $\mathfrak{a}$ is linked to an ideal generated by a Q-regular sequence. Cohen-Macaulay rings of codimension at most 2 are of this type, cf. [21, 3.3], so we obtain:

4.2. **Corollary.** *Let R be a Cohen-Macaulay local ring with* $\operatorname{codepth} R \leq 2$.
If $\operatorname{Ext}^i_R(D^R, R) = 0$ for all $i \in [1, \dim R + 1]$, then R is Gorenstein. □

5. Cohen-Macaulay rings with short reductions

In this section we consider higher dimensional versions of artinian rings whose maximal ideal has a low degree of nilpotence.

5.1. **Theorem.** *Let $(R, \mathfrak{m}, k)$ be a local ring containing an R-regular sequence $\boldsymbol{f}$ with the property that $\mathfrak{m}^3 \subseteq (\boldsymbol{f})$.*
If $\operatorname{Ext}^i_R(D^R, R) = 0$ for all $i \in [1, \dim R + 1]$, then R is Gorenstein.

Remark. In the special case when R is a k-algebra and $\operatorname{rank}_k(R)$ is finite, this is due to Asashiba [1]; a simplified proof is given by Asashiba and Hoshino [2]. A close reading of that proof shows that the hypothesis that R is a k-algebra can be avoided. We present that argument, referring to [1] and [2] whenever possible, and sketching modifications when necessary. A different proof of the theorem was obtained by Huneke, Şega, and Vraciu [15].

Proof. By Corollary 1.3 the ring R is Cohen Macaulay, so in view of 1.4 we may assume that D^R is an R-module, D. We may further assume $\mathfrak{m}^3 = 0$ and $\operatorname{Ext}^1_R(D, R) = 0$. Indeed, we get $\operatorname{Ext}^1_{R/(\boldsymbol{f})}(D/\boldsymbol{f}D, R/(\boldsymbol{f})) = 0$ by repeated applications of Lemma 3.3. Now note that $D/\boldsymbol{f}D$ is a dualizing module for $R/(\boldsymbol{f})$ by 1.5.6, and that R and $R/(\boldsymbol{f})$ are simultaneously Gorenstein.

For the rest of the proof we fix a free cover

$$0 \longrightarrow C \longrightarrow F \longrightarrow D \longrightarrow 0 \tag{5.1.1}$$

Step 1 ([1, 2.2]). The module C is indecomposable.

Step 2 ([1, 2.3]). There is an equality $\mathfrak{m}^2 = (0 : \mathfrak{m})_R$.

The proofs of the two steps above do not use the hypothesis that R is an algebra.

If M is an R-module, then we let $\ell(M)$ denote its length. The polynomial $[M] = \sum_{i \geqslant 0} \operatorname{rank}_k(\mathfrak{m}^i M/\mathfrak{m}^{i+1} M) t^i$ is called the *Hilbert series* of M.

Step 3 ([1, 2.4 and its proof]). There is an inequality $\ell(\mathfrak{m}^2) \leq 2$.
If equality holds, then $\ell(C/\mathfrak{m}C) = \ell(\mathfrak{m}/\mathfrak{m}^2)$.

The original proof of Step 3 proceeds through computations with Hilbert series, using Steps 2 and 3 for rank counts in short exact sequences of k-vector spaces; one only needs to replace the latter by counts of lengths of R-modules.

The assertion of the following step is proved without restrictions on R.

Step 4 ([2, 2.1]). If $\ell(D/\mathfrak{m}D)=2$, then there is an exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^R(D,D) \longrightarrow C\otimes_R D \longrightarrow \operatorname{Hom}_R(D,D) \tag{5.1.2}$$

If $\ell(M)$ is finite, then for every $x\in R$ a length count in the exact sequence

$$0 \longrightarrow (0:x)_M \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$

yields $\ell((0:x)_M)=\ell(M/xM)$. Such equalities are used in the proof below.

Step 5 ([2, Proofs of 2.2–2.4 and 3.4]). The ring R is Gorenstein.

If $\mathfrak{m}^2=0$, then $\mathfrak{m}C=0$, so the isomorphism $\operatorname{Ext}_R^1(D,R)\cong\operatorname{Ext}_R^2(C,R)$ induced by (5.1.1) yields $\operatorname{Ext}_R^2(k,R)=0$, and hence R is Gorenstein.

If $\ell(\mathfrak{m}^2)=1$, then $(0:\mathfrak{m})\cong k$ by Step 2, so R is Gorenstein.

By Step 3, for the rest of the proof we may assume $\ell(\mathfrak{m}^2)=2$ and $\ell(C/\mathfrak{m}C)=\ell(\mathfrak{m}/\mathfrak{m}^2)$. From 1.5.3 and Step 2 we get $\ell(D/\mathfrak{m}D)=2$. Since D is a faithfully injective R-module and $\operatorname{Hom}_R(D,D)\cong R$, we have

$$\operatorname{Hom}_R(\operatorname{Tor}_1^R(D,D),D)\cong\operatorname{Ext}_R^1(D,\operatorname{Hom}_R(D,D))\cong\operatorname{Ext}_R^1(D,R)$$

and hence $\operatorname{Tor}_1^R(D,D)=0$. Thus, from (5.1.2) we get an injection $C\otimes_R D\hookrightarrow R$. Since $\mathfrak{m}^2\neq 0$, we may choose an element $x\in\mathfrak{m}\smallsetminus(0:\mathfrak{m})_R$. We now have

$$\begin{aligned}
1+\ell(\mathfrak{m}/\mathfrak{m}^2) &= \ell(R)-2\\
&\geq \ell(R)-\ell(Rx)=\ell\big((0:x)_R\big)\\
&\geq \ell\big((0:x)_{C\otimes_R D}\big)=\ell\big((C\otimes_R D)/x(C\otimes_R D)\big)\\
&\geq \ell\big((C\otimes_R D)/\mathfrak{m}(C\otimes_R D)\big)=\ell(C/\mathfrak{m}C)\ell(D/\mathfrak{m}D)\\
&= 2\ell(\mathfrak{m}/\mathfrak{m}^2)
\end{aligned}$$

As a consequence, we get $\ell(\mathfrak{m}/\mathfrak{m}^2)\leq 1$, hence $\ell(\mathfrak{m}^2)\leq 1$, a contradiction. □

6. Artinian rings with Gorenstein deformations

Given a homomorphism of commutative rings $P\to Q$, let $D_P(Q)$ denote the Q-module $\operatorname{Hom}_P(Q,P)$ with the canonical action. The P-algebra Q is said to be *Frobenius* if Q is a finite free P-module and there is a Q-linear isomorphism $D_P(Q)\cong Q$. For every prime ideal $\mathfrak{q}$ of P we set $k(\mathfrak{q})=P_\mathfrak{q}/\mathfrak{q}P_\mathfrak{q}$.

6.1. **Theorem.** *Let $(P,\mathfrak{p},k)$ be a local ring and let $P\to Q$ be a homomorphism of commutative rings, such that Q is a finite free P-module, and there exists a prime ideal $\mathfrak{q}$ of P for which the ring $k(\mathfrak{q})\otimes_P Q$ is Gorenstein.*

If $R=Q/\mathfrak{p}Q$ satisfies $\operatorname{Ext}_R^1(D_k(R),R)=0$, then the ring R is Gorenstein and the P-algebra Q is Frobenius.

6.2. *Remark.* The ring R in the theorem is a finite k-algebra. For such rings, $X=\operatorname{Spec}R$ is finite and $R\cong\prod_{\mathfrak{m}\in X}R_\mathfrak{m}$ as k-algebras. This yields the first isomorphism of R-modules in the next formula, the second one comes from 1.5.1:

$$D_k(R)\cong\coprod_{\mathfrak{m}\in X}D_k(R_\mathfrak{m})\cong\coprod_{\mathfrak{m}\in X}D^{R_\mathfrak{m}}$$

For use in the proof we recall a couple of known facts, cf. e.g. [23, §14].

6.3. Let $P \to P'$ be a homomorphism of commutative rings, set $Q' = P' \otimes_P Q$, and let $P' \to Q'$ be the induced homomorphism. In the composition

$$Q' \otimes_Q D_P(Q) \longrightarrow P' \otimes_P D_P(Q) \longrightarrow D_{P'}(Q')$$

of canonical homomorphisms of Q'-modules the first arrow is always bijective. The second one is an isomorphism when Q is finite free over P.

6.4. **Lemma.** *If $(P, \mathfrak{p}, k)$ is a local ring and $P \to Q$ a homomorphism of commutative rings, then the following are equivalent.*

(i) *The P-algebra Q is Frobenius.*
(ii) *The P'-algebra $Q' = P' \otimes_P Q$ is Frobenius for every local ring P' and every homomomorphism of rings $P \to P'$.*
(iii) *The P-module Q is finite free and the ring $k(\mathfrak{q}) \otimes_P Q$ is Gorenstein for every $\mathfrak{q} \in \operatorname{Spec} P$.*
(iv) *The P-module Q is finite free and the ring $R = Q/\mathfrak{p}Q$ is Gorenstein.*

Proof. Consider first the special case $P = k$. The isomorphisms of Remark 6.2 then show that the ring R is Gorenstein if and only if $R_\mathfrak{m}$ is Gorenstein for every $\mathfrak{m}$, and the k-algebra R is Frobenius if and only if $D_k(R_\mathfrak{m}) \cong R_\mathfrak{m}$ for every $\mathfrak{m}$. The conditions at each $\mathfrak{m}$ are equivalent by 1.5.7, so (i) $\iff$ (iv) whenever P is a field.

Under the hypothesis of (i) the second map in 6.3 is bijective, so (ii) holds. When (ii) holds the algebra $k(\mathfrak{q}) \otimes_P Q$ over the field $k(\mathfrak{q})$ is Frobenius. By the special case this implies that the ring $k(\mathfrak{q}) \otimes_P Q$ is Gorenstein, which is the assertion of (iii). It is clear that (iii) implies (iv). If (iv) holds, then from Remark 6.2 and 1.5.7 we obtain an isomorphism of R-modules $k \otimes_Q D_P(Q) \cong R$. Nakayama's Lemma then yields an epimorphism of Q-modules $Q \to D_P(Q)$. Over P both modules are finite free and have the same rank, so this map is bijective. □

Proof of Theorem 6.1. By Lemma 6.4, the P-algebra Q is Frobenius if and only if the ring R is Gorenstein. The induced map $P/\mathfrak{q} \to Q/\mathfrak{q}Q$ turns $Q/\mathfrak{q}Q$ into a finite free $P/\mathfrak{q}$-module; due to the isomorphism of rings $(Q/\mathfrak{q}Q)/\mathfrak{p}(Q/\mathfrak{q}Q) \cong R$, the same lemma implies that R is Gorenstein if and only the $P/\mathfrak{q}$-algebra $Q/\mathfrak{q}Q$ is Frobenius.

Changing notation if necessary, for the rest of the proof we may assume that P is a domain with field of fractions K, and the ring $K \otimes_P Q$ is Gorenstein. Setting $D = D_P(Q)$, we now have isomorphisms of $K \otimes_P Q$-modules

$$K \otimes_P D \cong D_K(K \otimes_P Q) \cong K \otimes_P Q$$

given by 6.3 and Lemma 6.4. In view of Lemma 2.2, the desired isomorphism $D \cong Q$ will follow once we prove that the P-module $D \otimes_Q D$ is torsion-free.

Let $F \xrightarrow{\simeq} D$ be a resolution by free Q-modules, and let $k \xrightarrow{\simeq} J$ be a resolution by injective P-modules. There are then chains of relations

$$\begin{aligned} R \otimes_Q F &\cong (k \otimes_P Q) \otimes_Q F \cong k \otimes_P F \simeq k \otimes_P D \\ &\cong D_k(R) \end{aligned} \tag{6.4.1}$$

$$\begin{aligned} \operatorname{Hom}_P(D, J) &\simeq \operatorname{Hom}_P(D, k) \cong \operatorname{Hom}_k(k \otimes_P D, k) \cong \operatorname{Hom}_k(D_k(R), k) \\ &\cong R \end{aligned} \tag{6.4.2}$$

where the quasi-isomorphisms result from the freeness of D over P, cf. (A.2.3) and (A.2.1), the isomorphism $k \otimes_P D \cong D_k(R)$ comes from 6.3, and all other maps are canonical. Using (6.4.2) and (A.2.1) we get quasi-isomorphisms

$$\begin{aligned}\mathrm{Hom}_R(R \otimes_Q F, R) &\cong \mathrm{Hom}_Q(F, R) \simeq \mathrm{Hom}_Q(F, \mathrm{Hom}_P(D, J)) \\ &\cong \mathrm{Hom}_P(F \otimes_Q D, J)\end{aligned}$$

As (6.4.1) shows that $R \otimes_Q F$ is a free resolution of the R-module $D_k(R)$, Proposition B.1 applied to $G = F \otimes_Q D$ and J yields a strongly convergent spectral sequence

$${}^2\mathrm{E}_{p,q} = \mathrm{Ext}_P^{-p}(\mathrm{Tor}_{-q}^Q(D, D), k) \implies \mathrm{Ext}_R^{-p-q}(D_k(R), R)$$

with differentials ${}^r d_{p,q} \colon {}^r\mathrm{E}_{p,q} \to {}^r\mathrm{E}_{p+r,q+r-1}$. Since ${}^2\mathrm{E}_{p,q} = 0$ if $p > 0$ or $q > 0$, the spectral sequence defines an exact sequence

$$0 \longrightarrow \mathrm{Ext}_P^1(D \otimes_Q D, k) \longrightarrow \mathrm{Ext}_R^1(D_k(R), R)$$

which implies that $\mathrm{Ext}_P^1(D \otimes_Q D, k)$ vanishes. Because the P-module $D \otimes_Q D$ is finite, we conclude that it is actually free, and so *a fortiori* torsion-free. □

7. Rings of small codepth

In this section we prove the theorem below, after substantial preparation.

7.1. Theorem. *Let R be a local ring with* $\operatorname{codepth} R \leq 3$.
If $\mathrm{Ext}_R^i(D^R, R) = 0$ for all $i \gg 0$, then R is Gorenstein.

Let M, N be complexes with finite homology.

7.2.1. The R-modules $\mathrm{Tor}_i^R(M, N)$ and $\mathrm{Ext}_R^i(M, N)$ are finite for each $n \in \mathbb{Z}$, and vanish for all $n \ll 0$.

The preceding result is established by using standard arguments with spectral sequences. It ensures that the right hand sides of the equalities

$$\begin{aligned} I_R^M(t) &= \sum_{n\in\mathbb{Z}} \mathrm{rank}_k \big(\mathrm{Ext}_R^i(k, M)\big) t^i \\ P_M^R(t) &= \sum_{n\in\mathbb{Z}} \mathrm{rank}_k \big(\mathrm{Tor}_i^R(M, k)\big) t^i \end{aligned}$$

are formal Laurent series. They are called, respectively, the *Bass series* and the *Poincaré series* of M, and are invariant under quasi-isomorphisms by A.3.3, A.3.4.

7.2.2. If $F \to M$ is a semiprojective resolution and $m = \sup \mathrm{H}(M)$, then the module $M' = \mathrm{Coker}(\partial_{m+1}^F)$ satisfies $P_M^R(t) - t^m P_{M'}^R(t) \in \mathbb{Z}[t, t^{-1}]$.
Indeed, $\mathrm{Tor}_i^R(M, k) \cong \mathrm{Tor}_{i-m}^R(M', k)$ holds for each $i > m$ by Lemma B.5.

The next equality is due to Foxby, cf. [8, 4.1(a)] and [10, 15.18].

7.2.3. $I_R^{\mathbf{R}\mathrm{Hom}_R(M,N)}(t) = P_M^R(t) I_R^N(t)$

Once again, we focus upon the special case of dualizing complexes.

7.3. Set $d = \operatorname{depth} R$ and $D = D^R$.

7.3.1. $I_R^D(t) = t^d$ holds by 1.1.3.

We recall another fundamental property of dualizing complexes.

7.3.2. Set $M^\dagger = \mathbf{R}\mathrm{Hom}_R(M, D)$. The canonical map $M \to M^{\dagger\dagger}$ is then a quasi-isomorphism, cf. [10, 15.14]. For $M = R$ this yields $R \simeq \mathbf{R}\mathrm{Hom}_R(D, D)$.

From the formulas and quasi-isomorphisms above one now gets:

$$I_R^R(t) = I_R^{\mathbf{R}\mathrm{Hom}_R(D,D)}(t) = P_D^R(t) I_R^D(t) = P_D^R(t)t^d \tag{7.3.3}$$

$$I_R^{M^\dagger}(t) = P_M^R(t) I_R^D(t) = P_M^R(t)t^d \tag{7.3.4}$$

$$I_R^M(t) = I_R^{M^{\dagger\dagger}}(t) = P_{M^\dagger}^R(t) I_R^D(t) = P_{M^\dagger}^R(t)t^d \tag{7.3.5}$$

For the proof the theorem we need specific information on Poincaré series and Bass series of complexes. It suffices to have it for Poincaré series of modules:

7.4. **Proposition.** *If R is a local ring that has a dualizing complex and $c(t)$ is a polynomial in $\mathbb{Z}[t]$, then the following statements are equivalent:*

(1) $c(t)P_M^R(t) \in \mathbb{Z}[t]$ *for all finite R-modules M.*
(2) $c(t)P_M^R(t) \in \mathbb{Z}[t, t^{-1}]$ *for all complexes M with finite homology.*
(3) $c(t)I_R^M(t) \in \mathbb{Z}[t, t^{-1}]$ *for all complexes M with finite homology.*

Proof. Formulas (7.3.4) and (7.3.5) show that (2) and (3) are equivalent. Obviously, (2) implies (1), and the converse follows from 7.2.2. □

We need several results on Poincaré series of modules over rings of small codepth.

7.5. Let $(R, \mathfrak{m}, k)$ be a ring, such that $\operatorname{codim} R \le 3$.

The next result is proved by Avramov, Kustin and Miller [6, 6.1 and 5.18].

7.5.1. There exists a polynomial $d(t) \in \mathbb{Z}[t]$, such that $(1+t)d(t)P_M^R(t) \in \mathbb{Z}[t]$ for every finite R-module M and $(1+t)d(t)P_k^R(t) = (1+t)^{\operatorname{edim} R}$.

The table below gives a classification of the rings of codepth at most 3 obtained in [6, 2.1]; the values of $d(t)$ are computed by Avramov [3, 3.5].

7.5.2. If R is not complete intersection, then there exist integers l, m, p, q, r that determine the polynomial $d(t)$ from 7.5.1 through the following table:

type	codepth R	$d(t)$	restrictions
GO	2	$1 - t - lt^2$	$l \ge 1$
TE	3	$1 - t - lt^2 - (m - l - 3)t^3 - t^5$	$m > l + 1 \ge 3$
B	3	$1 - t - lt^2 - (m - l - 1)t^3 + t^4$	$m > l + 1 \ge 3$
$\mathbf{G}(r)$	3	$1 - t - lt^2 - (m - l)t^3 + t^4$	$m > l + 1 \ge 3$ $l + 1 \ge r \ge 2$
$\mathbf{H}(p, q)$	3	$1 - t - lt^2 - (m - l - p)t^3 + qt^4$	$m > l + 1 \ge 3$ $l \ge p \ge 0$ $m - l \ge q \ge 0$

As a consequence of [3, 3.6], we have:

7.5.3. If $d(t)$ has 1 as a root, then R is of type $\mathbf{H}(l, l-1)$ for some $l \ge 2$ and $d(t) = (1+t)(1-t)(1-t-(l-1)t^2)$.

Sun [25, Proof of 1.2, p. 61] provides additional information on the roots of $d(t)$.

7.5.4**.** If $d(t)$ has a real root r with $0<r<1$, then this root is simple.

In the proof of the theorem we use a technique developed by Şega [24]. In order to apply it, we need an extension of Sun's result.

7.6**. Lemma.** *Let R be a local ring with* $\operatorname{codim} R \leq 3$, *which is not complete intersection, and let $d(t)$ be the polynomial described in* 7.5.1.

If $p(t) \in \mathbb{Z}[t]$ is an irreducible polynomial with constant term 1 *and at least one negative coefficient, then $p(t)^2$ does not divide $d(t)$.*

Proof. We assume that our assertion fails, note that it implies $\deg p(t) \leq 2$, and obtain a contradiction for each type of rings described in 7.5.2.

GO. We must have $1-t-lt^2 = p(t)^2$, hence $l=-1/4$. This is absurd.

TE. Only $1+t$ can be a linear factor of $d(t)$, so $p(t)=1+at\pm t^2$ and $d(t) = (1+t)p(t)^2 = 1+\cdots+t^5$. However, by 7.5.2 the coefficient of t^5 is -1.

B or $\mathbf{G}(r)$. Only $1+t$ can be a linear factor of $d(t)$. Thus, $p(t)=1+at\pm t^2$ and $d(t)=p(t)^2 = 1+2at+\cdots$, yielding $a=-1/2$, a contradiction.

$\mathbf{H}(p,q)$. If $p(t)=1+at+bt^2$, then $d(t)=p(t)^2$, so we get a contradiction as above. If $p(t)=1-at$ with $a>1$, then $d(t)$ has a double real root $r=1/a<1$, contradicting 7.5.4. Finally, if $p(t)=1-t$, then $d(t)=(1+t)(1-t)(1-t-(l-1)t^2)$ by 7.5.3, so $1-t$ divides $1-t-(l-1)t^2$, hence $l=1$. This is impossible. $\square$

Proof of Theorem 7.1*.* Using 7.5.1 and 7.4, choose $d(t), r(t) \in \mathbb{Z}[t]$ such that

$$I_R^R(t) = \frac{r(t)}{(1+t)d(t)} \tag{7.6.1}$$

Set $D=D^R$ and $d=\operatorname{depth} R$. Our hypothesis means that $\mathrm{H}\,\mathbf{R}\mathrm{Hom}_R(D,R)$ is bounded. It is degreewise finite by 7.2.1, so using 7.2.3 and (7.3.3) we get

$$I_R^{\mathbf{R}\mathrm{Hom}_R(D,R)}(t) = P_D^R(t) I_R^R(t) = \frac{I_R^R(t)^2}{t^d} = \frac{r(t)^2}{t^d(1+t)^2 d(t)^2}$$

Referring again to 7.5.1 and 7.4, we now obtain

$$\frac{r(t)^2}{d(t)} = t^d(1+t)\left((1+t)d(t) I_R^{\mathbf{R}\mathrm{Hom}_R(D,R)}(t)\right) \in \mathbb{Z}[t,t^{-1}] \tag{7.6.2}$$

Let $s(t)$ denote the product of all the irreducible in $\mathbb{Z}[t]$ factors of $d(t)$ with constant term 1 and at least one negative coefficient, and set $q(t)=d(t)/s(t)$.

If R is complete intersection, then it is Gorenstein, so there is nothing to prove. Else, (7.6.2) and 7.6 imply that $s(t)$ divides $r(t)$ in $\mathbb{Z}[t,t^{-1}]$, so (7.6.1) yields

$$(1+t)q(t)I_R^R(t) = \frac{r(t)}{s(t)} \in \mathbb{Z}[t,t^{-1}]$$

Since $q(t)$ and $I_R^R(t)$ have non-negative coefficients, the last formula implies $I_R^R(t) \in \mathbb{Z}[t,t^{-1}]$, hence $\operatorname{id}_R R < \infty$. Thus, R is Gorenstein, as desired. $\square$

8. Golod rings

Let $(R,\mathfrak{m},k)$ be a local ring. Serre proved a coefficientwise inequality

$$P_k^R(t) \preccurlyeq \frac{(1+t)^{\operatorname{edim} R}}{1-\sum_{j=1}^{\infty} \operatorname{rank} \mathrm{H}_j(K^R)t^{j+1}}$$

of formal power series, where K^R denotes the Koszul complex on a minimal set of generators of $\mathfrak{m}$. If equality holds, then R is said to be a *Golod ring*.

The class of Golod rings is essentially disjoint from that of Gorenstein rings: if R belongs to their intersection, then R is a *hypersurface ring*, in the sense that its $\mathfrak{m}$-adic completion $\widehat{R}$ is the homomorphic image of a regular local ring by a principal ideal.

8.1. **Theorem.** *Let R be a Golod local ring.*
If $\operatorname{Ext}^i_R(D^R, R) = 0$ for all $i \gg 0$, then R is a hypersurface ring.

For the proof we recall a theorem of Jorgensen [18, 3.1]:

8.2. Let R be a Golod ring and L, M modules with finite homology.
If $\operatorname{Tor}^R_i(L, M) = 0$ for all $i \gg 0$, then L or M has finite projective dimension.

We need an extension of this result to complexes.

8.3. **Proposition.** *Let R be a Golod ring and L, M complexes with finite homology.*
If $\operatorname{Tor}^R_i(L, M) = 0$ for all $i \gg 0$, then $P^R_L(t)$ or $P^R_M(t)$ is a Laurent polynomial.

Proof. Set $l = \sup \mathrm{H}(L)$ and $m = \sup \mathrm{H}(M)$, choose semiprojective resolutions $E \to L$ and $F \to M$, then set $L' = \operatorname{Coker}(\partial^E_l)$ and $M' = \operatorname{Coker}(\partial^F_m)$. In view of Lemma B.5, our hypothesis yields $\operatorname{Tor}^R_i(L', M') = 0$ for all $i \gg 0$. By 8.2, this means that $P^R_{L'}(t)$ or $P^R_{M'}(t)$ is a polynomial, so 7.2.2 gives the desired assertion. □

Proof of Theorem 8.1. Set $D = D^R$ and choose a semiprojective resolution $F \to D$ with $\inf F = 0$. For all $i \gg 0$ we have $\mathrm{H}_i \operatorname{Hom}_R(F, R) = 0$ by hypothesis, hence $\mathrm{H}_i(F \otimes_R D) = 0$ by Proposition B.3(2), and thus $P^D_R(t) \in \mathbb{Z}[t, t^{-1}]$ by Proposition 8.3. Now (7.3.3) implies $I^R_R(t) \in \mathbb{Z}[t]$, hence $\operatorname{id}_R R < \infty$, that is, R is Gorenstein. As noted before the statement of the theorem, R is then a hypersurface ring. □

9. Relations with Tachikawa's conjecture

In this section A denotes a not necessarily commutative algebra of finite rank over a field k, and $D_k(A) = \operatorname{Hom}_k(A, k)$ has the canonical action of A on the left.

In [20] Nakayama proposed a now famous conjecture. The original statement was in terms of the injective resolution of A as a bimodule over itself, but Müller [19] proved that it is equivalent to the one-sided version below:

Conjecture (NC). If the left A-module A has an injective resolution in which each term is also projective, then A is selfinjective.

Considerable efforts notwithstanding, this still is one of the main open problems in the representation theory of Artin algebras. It has generated a significant body of research and has led to the study of several variants and to the formulation of equivalent forms, see Yamagata [28] for a general presentation of the subject. In particular, Tachikawa [26, Ch. 8] proved that the validity of Nakayama's conjecture is equivalent to that of both statements below for all k-algebras of finite rank.

Conjecture (TC1). If $\operatorname{Ext}^i_A(D_k(A), A) = 0$ for all $i > 0$, then A is selfinjective.

Conjecture (TC2). If the ring A is selfinjective and M is a finite A-module satisfying $\operatorname{Ext}^i_A(M, M) = 0$ for all $i > 0$, then M is projective.

Now we turn to the case when A is commutative. In view of Remark 6.2, for the conjectures above one may assume that A is local. The validity of Nakayama's Conjecture is then well-known and easy to see. However, both of Tachikawa's

Conjectures are open for commutative algebras. This apparent paradox is explained by the fact that the proof of the equivalence of Conjecture (NC) with Conjectures (TC1) and (TC2) involves intermediate non-commutative algebras.

Here are some instances in which Conjecture (TC1) is known to hold.

9.1. Let $(A, \mathfrak{r}, k)$ be a commutative local algebra of finite rank over k.

Each set of hypotheses below implies that the algebra A is selfinjective:

9.1.1. $\operatorname{edim} A \leq 2$ and $\operatorname{Ext}^i_A(D_k(A), A) = 0$ for $i = 1, 2$ (Hoshino [13]).

9.1.2. $\operatorname{rank}_k(A/xA) \leq 2$ for some $x \in \mathfrak{r}$ and $\operatorname{Ext}^1_A(D_k(A), A) = 0$ (Zeng [29]).

9.1.3. $\operatorname{rank}_k(A/xA) \leq 3$ for some $x \in \mathfrak{r}$ and $\operatorname{Ext}^i_A(D_k(A), A) = 0$ for $i = 1, 2$ (Asashiba and Hoshino [2]).

9.1.4. $\mathfrak{r}^3 = 0$ and $\operatorname{Ext}^1_A(D_k(A), A) = 0$ (Asashiba [1]).

The last result was extended in Theorem 5.1. Other results of this paper establish Conjecture (TC1) in new cases, some of them collected in the next theorem. We remark that Part (1) below improves 9.1.1 and implies 9.1.2.

9.2. **Theorem.** *A commutative local algebra A of finite rank over a field k is selfinjective whenever one of the following assumptions is satisfied.*

(1) $\operatorname{edim} A \leq 2$ *and* $\operatorname{Ext}^1_A(D_k(A), A) = 0$.
(2) $\operatorname{edim} A \leq 3$ *and* $\operatorname{Ext}^i_A(D_k(A), A) = 0$ *for all* $i \gg 0$.
(3) *A is Golod and* $\operatorname{Ext}^i_A(D_k(A), A) = 0$ *for all* $i \gg 0$.
(4) $A \cong Q/(\boldsymbol{f})$, *where Q is a reduced complete local ring, $\boldsymbol{f}$ is a Q-regular sequence, and* $\operatorname{Ext}^1_A(D_k(A), A) = 0$.

Proof. The first three parts are specializations of Corollary 4.2, Theorem 7.1, and Theorem 8.1, respectively. For (4), use Cohen's Structure Theorem to present Q as a residue ring of a regular local ring modulo a radical ideal. Such an ideal is generically complete intersection, so Theorem 3.1 applies. □

Appendix A. Constructions with complexes

In this appendix we describe terminology and notation that are used throughout the paper. Proofs of the basic results listed below can be found, for instance, in [12], [22], or [10]. We let R denote an associative ring over which modules have left actions, and M denote a complex of R-modules of the form

$$\cdots \longrightarrow M_{i+1} \xrightarrow{\partial^M_{i+1}} M_i \xrightarrow{\partial^M_i} M_{i-1} \longrightarrow \cdots$$

We write $b \in M$ to indicate that b belongs to M_i for some $i \in \mathbb{Z}$, and $|b| = n$ to state that b is in M_n. The n'th *shift* $\Sigma^n M$ of M is the complex with

$$(\Sigma^n M)_i = M_{i-n} \qquad \text{and} \qquad \partial^{\Sigma^n M}_i = (-1)^n \partial^M_{i-n}\,.$$

If N is a complex, then a *morphism* $\alpha\colon M \to N$ is a sequence of R-linear maps $\alpha_i\colon M_i \to N_i$, such that $\partial^N_i \alpha_i = \alpha_{i-1}\partial^M_i$ holds for all i. It is a *quasi-isomorphism* if $\mathrm{H}_i(\alpha)$ is bijective for each i; we use the symbol $\simeq$ to identify quasi-isomorphisms.

Set $\inf M = \inf\{i \mid M_i \neq 0\}$ and $\sup M = \sup\{i \mid M_i \neq 0\}$. The complex M is *bounded above* (respectively, *below*) if $\sup M$ (respectively, $\inf M$) is finite. It is *bounded* when both numbers are finite. It is *degreewise finite* if each R-module M_i is finite. If $\mathrm{H}(M)$ is bounded and degreewise finite, then M has *finite homology*.

A.1.1. We let $M_{<n}$ denote the subcomplex of M having $(M_{<n})_i = 0$ for $i \geq n$ and $(M_{<n})_i = M_i$ for $i < n$, and we set $M_{\geqslant n} = M/M_{<n}$.

A.1.2. We let $\tau_{\leqslant n}(M)$ denote the residue complex of M with $\tau_{\leqslant n}(M)_i = 0$ for $i > n$, $\tau_{\leqslant n}(M)_n = M_n/\operatorname{Im}(\partial^M_{n+1})$, and $\tau_{\leqslant n}(M)_i = M_i$ for $i < n$; the maps $\mathrm{H}_i(M) \to \mathrm{H}_i(\tau_{\leqslant n}(M))$ induced by the natural surjection are bijective for all $i \leq n$.

A.2. Let L be a complex of right R-modules and M, N be complexes of R-modules.

A complex $\operatorname{Hom}_R(M,N)$ is defined by the formulas

$$\operatorname{Hom}_R(M,N)_n = \prod_{j-i=n} \operatorname{Hom}_R(M_i, N_j) \qquad \partial(\beta) = \partial^N \circ \beta - (-1)^{|\beta|}\beta \circ \partial^M$$

and a complex $L \otimes_R M$ defined by the formulas

$$(L \otimes_R M)_n = \coprod_{h+i=n} L_h \otimes_R M_i \qquad \partial(a \otimes b) = \partial^L(a) \otimes b + (-1)^{|a|} a \otimes \partial^M(b)$$

If $\lambda\colon L \to L'$ is a quasi-isomorphism of complexes of right R-modules, $\mu\colon M \to M'$ is a quasi-isomorphism of complexes of R-modules, J is a bounded above complex of injective R-modules, and F is a bounded below complex of projective R-modules, then the maps below are quasi-isomorphisms.

$$\operatorname{Hom}_R(F,\mu)\colon \operatorname{Hom}_R(F,M) \longrightarrow \operatorname{Hom}_R(F,M') \tag{A.2.1}$$

$$\operatorname{Hom}_R(\mu,J)\colon \operatorname{Hom}_R(M',J) \longrightarrow \operatorname{Hom}_R(M,J) \tag{A.2.2}$$

$$\lambda \otimes_R F\colon L \otimes_R F \longrightarrow L' \otimes_R F \tag{A.2.3}$$

If, furthermore, $\phi\colon F \to F'$ is a quasi-isomorphism of bounded below complexes of projective R-modules, then the maps below are quasi-isomorphisms.

$$\operatorname{Hom}_R(\phi,M)\colon \operatorname{Hom}_R(F',M) \longrightarrow \operatorname{Hom}_R(F,M) \tag{A.2.4}$$

$$L \otimes_R \phi\colon L \otimes_R F \longrightarrow L \otimes_R F' \tag{A.2.5}$$

We use derived functors of complexes, for which we recall some basic notions.

A.3. If $\mathrm{H}(M)$ is bounded below, then there exists a quasi-isomorphism $F \to M$ with F a bounded below complex of projective modules, called a *semiprojective resolution* of M; it can be chosen with $\inf F = \inf \mathrm{H}(M)$. Any two semiprojective resolutions of M are homotopy equivalent, so the complexes $\operatorname{Hom}_R(F,N)$ and $L \otimes_R F$ are defined uniquely up to homotopy. The symbols $\mathbf{R}\operatorname{Hom}_R(M,N)$ and $L \otimes^{\mathbf{L}}_R M$ denote any complex in the corresponding homotopy class. One sets

$$\operatorname{Ext}^i_R(M,N) = \mathrm{H}_{-i}\, \mathbf{R}\operatorname{Hom}_R(M,N) \tag{A.3.1}$$

$$\operatorname{Tor}^R_i(L,N) = \mathrm{H}_i(L \otimes^{\mathbf{L}}_R N) \tag{A.3.2}$$

If $L \to L'$, $M \to M'$ and $N \to N'$ are quasi-isomorphisms, then so are

$$\mathbf{R}\operatorname{Hom}_R(M,N) \longrightarrow \mathbf{R}\operatorname{Hom}_R(M,N') \longleftarrow \mathbf{R}\operatorname{Hom}_R(M',N')$$

$$L \otimes^{\mathbf{L}}_R M \longrightarrow L' \otimes^{\mathbf{L}}_R M \longrightarrow L' \otimes^{\mathbf{L}}_R M'$$

by (A.2.1) and (A.2.4), respectively by (A.2.1) and (A.2.4). Thus, for all $i \in \mathbb{Z}$

$$\operatorname{Ext}^i_R(M,N) \cong \operatorname{Ext}^i_R(M',N') \tag{A.3.3}$$

$$\operatorname{Tor}^R_i(L,M) \cong \operatorname{Tor}^R_i(L',M') \tag{A.3.4}$$

A.3.5. We identify R-modules with complexes concentrated in degree zero. For modules the constructions above yield the classical derived functors.

Appendix B. Lemmas about complexes

The hypotheses here are the same as in Appendix A. In one form or another, the results collected below are probably known to (some) experts, but may not be easy to find in the literature. Furthermore, slight weakenings of the hypotheses may lead to false statements, so we are providing complete arguments.

We start with the computation of a classical spectral sequence.

B.1. **Proposition.** *If G is a complex of R-modules and J is a complex of injective R-modules with $\sup J = 0$, then there exists a spectral sequence having*

$$ {}^{2}\mathrm{E}_{p,q} = \mathrm{H}_p \operatorname{Hom}_R(\mathrm{H}_{-q}(G), J) \qquad \textit{and} \qquad {}^{2}d_{p,q} = \operatorname{Hom}_R(\mathrm{H}_{-q}(G), \partial^J_p) $$

and differentials acting by the pattern ${}^{r}d_{p,q} \colon {}^{r}\mathrm{E}_{p,q} \to {}^{r}\mathrm{E}_{p-r,q+r-1}$ for all $r \geq 2$.

If J or G is bounded below, then this sequence converges strongly to $\operatorname{Hom}_R(G, J)$, in the sense that for each $n \in \mathbb{Z}$ the group $\mathrm{H}_n \operatorname{Hom}_R(G, J)$ has a finite, exhaustive, and separated filtration, the component of degree q of the associated graded group is isomorphic to ${}^{\infty}\mathrm{E}_{n-q,q}$, and ${}^{r}\mathrm{E}_{p,q} = {}^{\infty}\mathrm{E}_{p,q}$ for all $r \gg 0$ and all $(p, q) \in \mathbb{Z} \times \mathbb{Z}$.

Proof. The filtration of J by its subcomplexes $J_{\leqslant p}$ induces a filtration

$$ \mathrm{F}_p \operatorname{Hom}_R(G, J) = \operatorname{Im}\big(\operatorname{Hom}_R(G, J_{\leqslant p}) \to \operatorname{Hom}_R(G, J)\big) $$

of $\operatorname{Hom}_R(G, J)$. The differentials ${}^{r}d_{p,q}$ in the resulting spectral sequence ${}^{r}\mathrm{E}_{p,q}$ have the desired pattern for all $r \geq 0$ and all $(p, q) \in \mathbb{Z} \times \mathbb{Z}$. The isomorphisms

$$ {}^{0}\mathrm{E}_{p,q} = \Big(\frac{\mathrm{F}_p \operatorname{Hom}_R(G, J)}{\mathrm{F}_{p-1} \operatorname{Hom}_R(G, J)}\Big)_{p+q} \cong \operatorname{Hom}_R(G_{-q}, J_p) $$

take ${}^{0}d_{p,q}$ to $(-1)^q \operatorname{Hom}_R(\partial^G_{-q+1}, J_p)$. As each J_p is injective, the isomorphisms

$$ {}^{1}\mathrm{E}_{p,q} = \mathrm{H}_{-q} \operatorname{Hom}_R(G, J_p) \cong \operatorname{Hom}_R(\mathrm{H}_{-q}(G), J_p) $$

transfer ${}^{1}d_{p,q}$ into $\operatorname{Hom}_R(\mathrm{H}_{-q}(G), \partial^J_p)$. Thus, they lead to isomorphisms

$$ {}^{2}\mathrm{E}_{p,q} \cong \mathrm{H}_p \operatorname{Hom}_R(\mathrm{H}_{-q}(G), J) $$

that take ${}^{2}d_{p,q}$ to $\operatorname{Hom}_R(\mathrm{H}_{-q}(G), \partial^J_p)$.

If J or G is bounded below, then the filtration of $\operatorname{Hom}_R(G, J)_n$ by its subgroups $(\mathrm{F}_p \operatorname{Hom}_R(G, J))_n$ is finite for each $n \in \mathbb{Z}$; this implies strong convergence. □

Next comes a small variation on [5, 4.4].

B.2. **Lemma.** *If E is a complex of right R-modules, G denotes the complex of R-modules $\operatorname{Hom}_R(E, R)$, and J is a complex of R-modules, then the map*

$$ \begin{aligned} \theta^{EJ} \colon E \otimes_R J &\longrightarrow \operatorname{Hom}_R(G, J) \\ x \otimes y &\longmapsto \big(\gamma \mapsto (-1)^{|\gamma||y|}\gamma(x)y\big) \end{aligned} $$

is a morphism of complexes. It is bijective if J is bounded above, E is bounded below, and each right module E_h is finite projective.

Proof. It is easily verified that θ^{EJ} is a morphism of complexes.

By definition, $\operatorname{Hom}_R(G, J)_n = \prod_{i-(-h)=n} \operatorname{Hom}_R(G_{-h}, J_i)$ for each $n \in \mathbb{Z}$. The boundedness hypotheses imply that the abelian groups $\operatorname{Hom}_R(G_{-h}, J_i)$ are trivial for almost all pairs (h, i) satisfying $i - (-h) = n$, so the product of these groups is equal to their coproduct. The homomorphism $\theta^{EJ}_n \colon (E \otimes_R J)_n \to \operatorname{Hom}_R(G, J)_n$ is thus the coproduct of the canonical evaluation maps $E_h \otimes_R J_i \to \operatorname{Hom}_R(G_{-h}, J_i)$, each one multiplied by $(-1)^{ni}$. For every $h \in \mathbb{Z}$ the R-module $G_{-h} = \operatorname{Hom}_R(E_h, R)$ is finite projective, all the evaluation homomorphisms are bijective. $\square$

Now we are ready to produce some canonical isomorphisms. Our approach was suggested by results in an early version of a paper of Huneke and Leuschke [14].

When N is a right R-module $N^* = \operatorname{Hom}_R(N, R)$ carries the standard left action.

B.3. **Proposition.** *Let E be a complex of finite projective right R-modules with* $\inf E = 0$*, let J be a complex of R-modules, and form the composition of morphisms*

$$\vartheta^{EJ} \colon E \otimes_R J \xrightarrow{\theta^{EJ}} \operatorname{Hom}_R(\operatorname{Hom}_R(E, R), J) \xrightarrow{\operatorname{Hom}_R(\operatorname{Hom}_R(\epsilon,R),J)} \operatorname{Hom}_R(N^*, J)$$

where $N = \mathrm{H}_0(E)$, the map $\epsilon\colon E \to N$ is canonical, and θ^{EJ} is from Lemma B.2.

When J is bounded and each R-module J_i is injective the following hold.

(1) *If $\mathrm{H}_{-i}\operatorname{Hom}_R(E, R) = 0$ for all $i > 0$, then ϑ^{EJ} is a quasi-isomorphism.*
(2) *If $\mathrm{H}_{-i}\operatorname{Hom}_R(E, R) = 0$ for all $i \gg 0$, then $\mathrm{H}_i(E \otimes_R J) = 0$ for all $i \gg 0$.*
(3) *If $\sup J = 0$ and there is number $m \geq 1$ such that $\mathrm{H}_{-i}\operatorname{Hom}_R(E, R) = 0$ for all $i \in [1, m]$, then for every $i \leq m + \inf J$ there is an isomorphism*

$$\mathrm{H}_i(\vartheta^{EJ})\colon\ \mathrm{H}_i(E \otimes_R J) \xrightarrow{\cong} \mathrm{H}_i \operatorname{Hom}_R(N^*, J)$$

Proof. For $G = \operatorname{Hom}_R(E, R)$ Lemma B.2 yields an isomorphism of complexes

$$\theta^{EJ} \colon E \otimes_R J \xrightarrow{\cong} \operatorname{Hom}_R(G, J)$$

(1) By hypothesis, $\operatorname{Hom}_R(\epsilon, R)\colon N^* \to G$ is a quasi-isomorphism, hence by A.2.2 so is $\operatorname{Hom}_R(\operatorname{Hom}_R(\epsilon, R), J)$; as θ^{EJ} is bijective, we are done.

(2) The hypothesis means that $\mathrm{H}(G)$ is bounded. The spectral sequence of Proposition B.1 shows that so is $\mathrm{H}\operatorname{Hom}_R(G, J)$, which is isomorphic to $\mathrm{H}(E \otimes_R J)$.

(3) Proposition B.1 yields a strongly convergent spectral sequence

$${}^2\mathrm{E}_{p,q} = \mathrm{H}_p \operatorname{Hom}_R(\mathrm{H}_{-q}(G), J) \implies \mathrm{H}_{p+q} \operatorname{Hom}_R(G, J)$$

Note that $\inf J = -n$ for some integer $n \geq 0$. Because $G_{-q} = 0$ for all $q < 0$, and $\mathrm{H}_{-q}(G) = 0$ for all $q \in [1, m]$, we get ${}^2\mathrm{E}_{p,q} = 0$ for all pairs (p, q) with $p \notin [-n, 0]$ or with $q \leq m$, unless $q = 0$. It follows that if $p + q \leq m - n$, then

$${}^\infty\mathrm{E}_{p,q} = {}^2\mathrm{E}_{p,q} \cong \begin{cases} \mathrm{H}_p \operatorname{Hom}_R(N^*, J) & \text{when } q = 0 \\ 0 & \text{when } q \neq 0 \end{cases}$$

As a consequence, the edge map $\mathrm{H}_p \operatorname{Hom}_R(G, J) \to {}^2\mathrm{E}_{p,0}$ is bijective for $p \leq m-n$. It follows from the construction of the spectral sequence in Proposition B.1 that the isomorphism above takes this edge map to $\mathrm{H}_p \operatorname{Hom}_R(\operatorname{Hom}_R(\epsilon, R), J)$. $\square$

The next corollary contains as special cases [11, 2.3] and part of [11, 2.1].

B.4. **Corollary.** *Let L be a right R-module that has a projective resolution E by finite projective right R-modules, and let M be an R-module with* $\mathrm{id}_R M = n < \infty$*.*

If $m \geq 1$ is a number such that $\operatorname{Ext}^i_R(L, R) = 0$ *for all* $i \in [1, m]$*, then*

(1) $\operatorname{Ext}^i_R(L^*, M) = 0$ *for all* $i > \max\{0, n - m - 1\}$*.*
(2) $\operatorname{Tor}^R_i(L, M) = 0$ *for all* $i \in [1, m]$*.*
(3) *If $m \geq n$, then* $L \otimes_R M \cong \operatorname{Hom}_R(L^*, M)$*.*

Proof. Let $M \to J$ be an injective resolution with $J_i = 0$ for $i < -n$. The last part of the proposition yields for all $i \leq m - n$ isomorphisms

$$\operatorname{Tor}_i^R(L, M) \cong \operatorname{Ext}_R^{-i}(L^*, M)$$

The desired assertions follow, since $\operatorname{Tor}_j^R(L, M) = 0 = \operatorname{Ext}_R^j(L^*, M)$ for $j < 0$. □

Here is a version of the familiar degree-shifting procedure.

B.5. **Lemma.** *Let L be a complex of right modules with $\sup \mathrm{H}(L) = l < \infty$ and M a complex with $\sup \mathrm{H}(M) = m < \infty$. If $E \to L$ and $F \to M$ are semiprojective resolutions, then for the R-modules $L' = \operatorname{Coker}(\partial_l^E)$ and $M' = \operatorname{Coker}(\partial_m^F)$ there are isomorphisms $\operatorname{Tor}_i^R(L, M) \cong \operatorname{Tor}_{i-l-m}^R(L', M')$ for all $i > l + m$.*

Proof. The quasi-isomorphism $L \simeq \tau_{\leqslant l}(L)$, cf. A.1.2, induces by (A.2.3) a quasi-isomorphism $L \otimes_R F_{<m} \simeq \tau_{\leqslant l}(L) \otimes_R (F_{<m})$. As a consequence, we get

$$\begin{aligned} \sup \mathrm{H}(L \otimes_R (F_{<m})) &= \sup \mathrm{H}(\tau_{\leqslant l}(L) \otimes_R (F_{<m})) \\ &\leq \sup(\tau_{\leqslant l}(L) \otimes_R (F_{<m})) \\ &= l + m - 1 \end{aligned}$$

The homology exact sequence of the exact sequence of complexes

$$0 \to L \otimes_R (F_{<m}) \longrightarrow L \otimes_R F \longrightarrow L \otimes_R F_{\geqslant m} \to 0$$

now yields isomorphisms $\mathrm{H}_i(L \otimes_R F) \cong \mathrm{H}_i(L \otimes_R F_{\geqslant m})$ for all $i > l + m$. Since $\Sigma^{-m} F$ is a projective resolution of M', these isomorphisms can be rewritten as $\operatorname{Tor}_i^R(L, M) \cong \operatorname{Tor}_{i-m}^R(L, M')$ for all $i > l + m$. Similar arguments applied to the resolution $E \to L$ yield $\operatorname{Tor}_{i-m}^R(L, M') \cong \operatorname{Tor}_{i-l-m}^R(L', M')$ for all $i > l + m$. □

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